

Sensitivity and robustness to model risk in Lévy and jump-diffusion setting

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The market models rely on many choices, the structure of the model, the interpretation of the distribution of the noise, the number and type of parameters included. Different traders may have different perceptions of the market data and modeling. Recently, the dynamics of asset prices seem to be well modeled by Lévy noise and most of current research in mathematical finance is focused around this class (see e.g. Cont and Tankov [23]). These models generalize the classical continuous type models based on the Brownian motion to include possible jumps of the market prices. The jumps may also be of infinitely small size and occur with high intensity. Furthermore, it is a philosophical question whether asset prices are driven by pure-jump noise, or if there is a diffusion in the non-Gaussian dynamics (see e.g. Eberlein and Keller [30] for a discussion). From a statistical point of view it may be very hard to determine whether a model should have a diffusion term or not.

This thesis deals with the robustness of sensitivity analysis to the approximation of the underlying modeling noise and the study of the consequences of the choice of the model in the risk analysis and the hedging of financial claims.

1.1 Small jump approximation of Lévy noise

From the point of view of robustness to model choice, our point of departure is the paper of Asmussen and Rosinski [3], where it is proven that the small jumps of a Lévy process $L(t)_{t \geq 0}$ can be approximated by a Brownian motion scaled with the standard deviation of the small jumps, that is,

$$L(t) \approx \sigma(\varepsilon)B(t) + N^\varepsilon(t), \quad (1.1)$$

where $N^\varepsilon(t)_{t \geq 0}$ is a Lévy process with jumps bigger than ε and $B(t)_{t \geq 0}$ is an independent Brownian motion. The function $\sigma(\varepsilon)$ is the standard deviation of the jumps smaller than ε of the Lévy process, which can be computed as the integral of z^2 with respect to the Lévy measure in a ball of radius ε . Obviously, $\sigma(\varepsilon)$ tends to zero with ε . In fact, this approximating Lévy process converges in distribution to the original one.

In the case when we have a multidimensional Lévy diffusion, one can approximate the small jumps by a continuous martingale with appropriately scaled variance.

This is an important consideration also from the modeling point of view, in fact it is very hard from the point of view of statistics, if at all possible, to decide which model for price dynamics is best between one where the small variations in the asset dynamics come from a jump process with infinite activity or from a continuous martingale. Notice that the two models have the same variance. Moreover, in practice, it may be difficult to simulate from a Lévy-diffusion directly. One may approximate the small jump part by an appropriate scaled continuous part and observe that the remaining process is a compound Poisson. These are simple to simulate on a computer and the approximating dynamics may be discretized.

Based on this approximation, which is popular when simulating the paths of different Lévy processes like for instance the normal inverse Gaussian (see Rydberg [57]), we first, investigate the convergence of option prices after a change of measure (from the results of Asmussen and Rosinski [3], we know that the respective option prices converge when ε goes to zero). Moreover, we investigate the computation of the sensitivities derived from the models we considered. We also study the robustness of the sensitivities, we focus on the sensitivity with respect to the initial condition known as delta.

1.2 Robustness of option prices

In incomplete markets, not every contingent claim can be replicated by a self-financing strategy. Instead of eliminating the risk by a perfect hedge, the issuer can adopt a partial hedging strategy according to some optimality criteria minimizing the risk exposure, and in the end bearing some of the risk (see e.g. Cont and Tankov [23] for more about pricing and hedging in incomplete markets).

In Chapter 3, we consider an incomplete market where stock price fluctuations are modeled by a geometric Lévy process $S(t) = S(0)\exp(L(t))$, with $L(t)_{t \geq 0}$ being a Lévy process under the physical measure. Considering the approximation in equation (1.1), we can obtain another model for the dynamics of the stock price. The question is whether the option prices and their Greeks, under a risk-neutral equivalent martingale measure, converge. In this thesis, we show that this is indeed the case for the most popular choices of equivalent martingale measures. The problem we are facing here is that the choice of pricing measure is dependent on the approximation.

Due to market incompleteness for these models, there will exist infinitely many equivalent measures under which the discounted price processes are martingales. Gerber and Shiu [42, 43] proposed the *Esscher transform* as a potential pricing measure for Lévy models (see also Bühlmann et al. [15]). They explain their choice by modeling investor preferences by a power utility function and prove that in this case the investor's price when issuing an option is given by the expected discounted payoff computed with respect to the Esscher measure.

Another popular choice is the *minimal entropy martingale measure*, which is the probability of having minimum relative entropy with respect to the market probability (see Goll and Rüsendorf [41]). Fujiwara and Miyahara [35] show that the minimal entropy martingale measure is given by an Esscher transformation for exponential Lévy models of the stock price dynamics.

The *minimal martingale measure*, first introduced by Föllmer and Sondermann [36] for martingales and later extended to the general semimartingale case by Föllmer and Schweizer [37], is defined via locally risk minimizing hedging strategies. One considers strategies which have a cost $C > 0$. It turns out that the value process of a strategy that is minimizing locally the residual risk is given by the conditional expectation of the option's payoff under the minimal martingale measure. One drawback with this approach is the fact that one has to work with strategies which are not self-financing. If one prefers to avoid intermediate costs or unplanned income, a second idea is to insist on self-financing strategies that minimize the terminal hedging error in the mean-square sense. The *mean-variance optimal measure* is then used to calculate mean-variance optimal strategies (see Schweizer [64]).

Considering each of these equivalent measures, we prove that the option prices in the approximating model for the underlying stock converge to the prices derived on the stock dynamics modeled via the corresponding infinite activity Lévy process. By our results we have robustness in option prices and their Greeks with respect to this modelling choice. Moreover, in numerical procedures such an approximation comes in handy, since stability results are crucial for defending the approximation from an application point of view.

1.3 Computation of the delta and robustness

The delta of an option is defined as the sensitivity of the option price with respect to the state of the underlying asset. In mathematical terms, this is given as the derivative of $\mathbb{E}[f(X^x(T))]$ with respect to $X(0) = x$, where $X^x(t)_{t \geq 0}$ is the price dynamics of the underlying asset. In complete markets, the delta is known to be the number of assets $X^x(T)$ to hold in a self-financing portfolio exactly replicating the option $f(X(T))$. This is known as the delta-hedge. This is important also in incomplete markets for the construction of partial hedges (see for instance Cont and Tankov [23] for more on incomplete markets and partial hedging). Moreover, the delta being a sensitivity evaluation of the option price to variations in the underlying, it gives important information of the risk associated to an investment in the option both in complete and incomplete markets.

There are several methods for the computation of the delta one of them is the *differentiation method* which simply computes the derivative of the expectation by exchanging differentiation and integration and thus computing the expectation of the derivative of the payoff. The basic assumption of this technique is the differentiability of the payoff function which is not always holding. For example, for a plain vanilla call or put option, the payoff has a kink at the strike price. Although you get the right expectation by formally differentiating, the method becomes numerically very slow when applying the Monte Carlo simulation to evaluate the resulting expectation. For other options, like the digitals, one cannot find the derivative of the payoff function, ruling out this technique. The numerical counterpart to this method is *finite differencing*. Here one perturbs the option price slightly to calculate the finite difference which is the numerical approximation of the derivative. The computation of the Greek is then carried out via the computation of two similar expectations, which can be efficiently done by Monte Carlo methods if one applies the technique of common random numbers and the payoff function is differen-

tible. However, for non-differentiable payoffs, the method becomes very inefficient in the sense of slow Monte Carlo convergence. One way to deal with this problem is to consider either the *density method* or the *Malliavin method*. Both approaches have the advantage of not differentiating the payoff function f of the option and in both cases we have, for the delta, a formula of the type

$$\Delta = \mathbb{E}[f(X^x(T))\pi],$$

where π is a random variable called weight. This expectation functional is suitable for Monte-Carlo simulation.

The *density method* is based on the knowledge of the probability density of the price process. By moving the dependency of the initial price process to the density, one may differentiate this rather than the payoff function. The result is an expectation function of the payoff function times the logarithmic derivative of the density evaluated at the spot price at maturity of the option. We refer to Broadie and Glasserman [13] for more on this method.

The *Malliavin method* is based on an integration by parts formula to derive an expression for the delta not involving any differentiation of the payoff function. This approach is introduced by Fournié *et al.* [34] and it is well-developed for the Brownian case, but for jump diffusion models, it is not straightforwardly generalized due to the lack of a classical chain rule. Davis and Johansson [24] propose to use the Malliavin approach *only* on the Wiener term in the jump-diffusion dynamics where the jump part is driven by a Poisson process.

In Chapter 4, we introduce the *conditional density method* to compute the delta written in models driven by Lévy process. This method allows some flexibility in the computation when dealing with Lévy models not of Brownian nature. The conditional density method relies on the observation that we may use conditioning in order to separate out differentiable density in the expectation function. More precisely, if we have a random variable which may be represented as a sum of two independent random variable, where one possesses a differentiable density, we may use conditional expectation and the “classical” density approach to move the differentiation to this density. We recall from the Lévy-Kintchine representation of Lévy processes that any Lévy process can be represented as a pure-jump process and an independent drifted Brownian component. The application of the conditional density method provides *different* weights than the density method. The fact that the weights are not unique is well-known, as this appears also by application to other methods of computations, e.g. the so-called Malliavin methods. We stress that the delta is in any case the same, only the computation method is different. It is well-known that the density method provides an expression for the delta which has minimal variance. This is the meaning of *optimality* for weights. The weights derived by the conditional density method are not optimal.

From the point of view of robustness to model choice and considering options written on a Lévy process which has small jumps and options written on the approximation given by (1.1) we prove that the respective deltas converge when ε goes to 0. In itself this is maybe not a priori surprising but it turns out that for pure-jump Lévy processes one obtain weights for the approximating model which explode when ε tends to zero. Hence, the random variable inside the expectation diverges. However, due to an independence

property in the limit which is not found in the classical setting of the density method, the delta converges anyhow. However, the variance of the expression explodes, which in turn implies that the weights are highly inefficient from a Monte Carlo point of view. The same problem does not occur for Lévy processes having a continuous martingale part. Hence, we conclude that even though the delta is robust towards these approximations, the resulting expressions for the deltas may become inefficient for practical simulation, at least in the pure-jump case. We study numerical examples discussing this problem. Also, we provide convergence rates for the approximating deltas.

In the case where the price of the underlying is modeled by jump-diffusions, the density of the continuous martingale part is not always known and hence the use of the conditional density method is not applicable. Therefore, to derive expressions for the delta, we use a Malliavin calculus approach.

In Chapter 5, we extend the idea of Davis and Johansson [24] for the computation of the delta to substantially more general jump-diffusion processes. Our results are based on the Malliavin calculus for jump processes developed by Solé, Utzet, and Vives [65] and Di Nunno [25] (see also Di Nunno, Øksendal, and Proske [26]). We demonstrate that one may use the Malliavin approach also in cases where there are no continuous martingale components in the jump-diffusion dynamics. In this situation, one can *approximate* the small jumps by a continuous martingale with appropriately scaled variance (see Proposition 5.2.1, Chapter 5) and it turns out that the derived delta based on this approximation is close to the true one (see Theorem 5.3.1, Chapter 5). Hence, the Malliavin approach can be used to derive approximating deltas in the case when we face a jump-diffusion model without any continuous martingale part present in the dynamics. Our results show that, for what option pricing is concerned, the difference is for practical purposes negligible and the deltas are robust towards small changes in the underlying dynamics. We remark that, similar to the conditional density method, there are different ways of applying the Malliavin method, with the result that there are several equivalent expressions of the same delta.

We also deal with another method for computing the deltas, this is the *Fourier approach*. This method, in fact, has the advantage that it can be directly applied to models with or without continuous martingale part. However, it is actually difficult to implement since it requires an explicit solution of the stochastic differential equation describing the first variation process (see (5.24)). Within this methodology we again study the expressions for the deltas and prove robustness. Some examples are also detailed.

1.4 Application to power and commodity market

Most of the popular spot price dynamics applied in commodity and power markets are so-called multi-factor models. For a market like electricity, it is reasonable to have factors accounting for the spike behavior observed in the spot price series, whereas other factors model the price evolution when the market is in stable conditions. Commodity prices are often said to be mean-reverting, since the law of supply and demand will push prices back if they deviate too much from a mean level. On the other hand, this mean level may be significantly influenced by the resource situation of a commodity (oil say), and

thereby also stochastic. Hence, one often encounters two-factor models, essentially trying to capture mean-reverting prices around a randomly fluctuating mean. Typical models are the Schwartz-Smith dynamics applied to commodities or the multi-factor model of Benth, Kallsen, and Meyer-Brandis [9] developed for electricity spot prices.

In Chapter 6 we are concerned with the Greeks of options written on such multi-factor dynamics. There exists options in commodity markets which are written on the spot and forward price and to understand the risk involved in option investments one needs to calculate the Greeks. We shall concentrate on the delta and gamma of an option. The gamma is the second derivative of the price with respect to the current spot price.

We apply the *conditional density method*. The approach is simple: one applies the conditional expectation with respect to one of the factors and then uses the standard density method approach. To make this work, we need to have accessible the density of the factor we choose to condition on. As it turns out, the conditional density method is particularly useful for deriving the Greeks in the case of multi-factor models.

The conclusion of our findings is that as long as there is one component with a density in the spot price dynamics and as long as methods for simulating the spot price exist, one can compute the delta and gamma by simply Monte Carlo simulation of the spot. Furthermore, the delta and gamma are both expressible in terms of the price of an option with payoff equal to the original option's payoff times the density evaluated at the value of the component at exercise.

We illustrate our findings by several examples where we also perform a numerical analysis of efficiency and practical tractability. In particular, we look at a model without any Gaussian component, but with a known stationary distribution. We analyze how one can approximate the delta by calculating the corresponding expectation based on the stationary density instead. Our numerical experiments show that our conditional density method provides expressions which are highly tractable and easily implementable for numerical computation of the Greeks of options on multi-factor models.

There exist other methods, for instance, based on the Malliavin derivative (see Lions et al [34] and Benth, Dahl, and Karlsen [8] for an application to energy) or by numerical solution of the partial (integro-) differential equations associated to the option price (see Tankov, Cont, and Voltchkova [63]). Note that our expressions for the delta and gamma will themselves be solutions of partial (integro-) differential equations. Also, in our set-up, if possible, the Malliavin method will yield the same expressions and therefore not provide any new insight. However, when dealing with path-dependent options, the Malliavin approach would be fruitful.

1.5 Application to stochastic volatility models

In the Black-Scholes option pricing theory, asset prices are modeled by a geometric Brownian motion with a constant volatility parameter. However, it has been observed that the implied volatility depends on the strike price and the expiration date implying the so-called "volatility smile". This shows the limitations of the Black-Scholes model. An alternative is to model the market price processes by jumps and stochastic volatility. These models seem to be more robust and closer to reality. In fact, the market is usually

incomplete and one can't hedge away all the risks.

In Chapter 7 we aim to compute the delta of the option written in multidimensional jump-diffusions. We use the same Malliavin approach as in Chapter 5 and we apply this to the computation of the delta for stochastic volatility models. We study the robustness of the price processes when we approximate the multidimensional small jumps by an appropriately scaled martingale and we show that both the price processes and the deltas of the two models converge.

The Asian option has been widely studied. Caramellino and Marchisio [19] and El-Kathib and Privault [28] studied representation formulas for the delta of Asian options using a Malliavin calculus. They considered models in which the jump part is driven by a Poisson process. In this paper, we derive an expression for the delta of Asian options written in more general-jump diffusion processes and we prove the robustness of the option price and its delta.

As an application, we consider a general stochastic volatility model. That is we model the price process by a stochastic differential equation in which the volatility $\sigma(t)_{t \geq 0}$ is a function of another process. In that case, $\sigma(t) = f(Y(t))$, where f is a smooth, positive, and increasing function and the dynamics of $Y(t)_{t \geq 0}$ form a stochastic differential equation driven by a continuous part and a jump part. The continuous part of the process $Y(t)_{t \geq 0}$ is correlated with the Brownian motion of the underlying's price. Cass and Friz [18] compute the delta for stochastic volatility models using the Bismut-ElWorthy-Li formula. In this thesis, to compute the delta we consider a Malliavin derivative with respect to the Wiener term of the underlying's price. The weights we obtain involve the stochastic volatility. As an example we consider the Heston model (see Heston [44]) in which the function f is the square root of the process $Y(t)_{t \geq 0}$ and the process $Y(t)_{t \geq 0}$ is a continuous mean-reverting process. We also consider a Heston model which has jumps in the volatility (see Matytsin [52] and Sepp [58]). These models have nice properties, they directly model the observed random behavior of market volatility and allow to reproduce more realistic returns distributions, in particular, thicker than log-normal tails. They also provide a closed form solution for European options making it more tractable and easier to implement than other stochastic volatility models.

Moreover, we consider the BN-S model, introduced by Barndorff-Nielsen and Shephard [5], in which the stochastic variance of log-returns is constructed via a mean-reverting, stationary process of the Ornstein-Uhlenbeck type driven by a subordinator. That is the variance of the price process is given by

$$dY(t) = -\lambda Y(t)dt + dZ(t),$$

where $\lambda > 0$ and $Z(t)_{t \geq 0}$ is a subordinator. In applications, the term λ will be approximated. In this thesis, we approximate the term λ by λ_ε and we investigate the robustness of the model and of the associated option price. As the market is incomplete, we consider a structure preserving class of equivalent martingale measures introduced by Nicolato and Venardos [53] and we prove the convergence of the option price after a change of measure in this class. For the computation of the delta written in such models, we refer to Benth, Groth, and Wallin [12].

1.6 Conclusion

In this thesis we consider the problem of robustness of the option price and the sensitivity parameters to model choice. Considering exponential Lévy models, we prove the robustness of option price after a change of measure. The measures that we considered are selected among the most popular choices of risk neutral equivalent martingale measures. Moreover, we prove the robustness of the sensitivity parameter delta of options written in such models. Dealing with Lévy models, we introduce the conditional density method. The latter provides the existence of a density of an independent variable in the underlying model. We also derive expressions for the delta of options written in a general jump-diffusion model using the Malliavin calculus. We apply our methods for the computation of the delta to power and commodity market models as well as to stochastic volatility models and we illustrate our results with several numerical examples.

1.7 Structure of the thesis

The thesis is organized as follows. In Chapter 2, we introduce some notations and we present different methods for the computation of the delta in the continuous case. Chapter 3 (extracted from the article "A note on convergence of option prices and their Greeks for Lévy models" by Fred Espen Benth, Giulia Di Nunno, and Asma Khedher, available as *E-print*, No. 18, November (2010), Department of Mathematics, University of Oslo, Norway, submitted for publication) is dedicated to the study of the problem of robustness of prices to model choice under change of measure. Chapter 4 (extracted from the article "Lévy models robustness and sensitivity" by Fred Espen Benth, Giulia Di Nunno, and Asma Khedher, published in QP-PQ: Quantum Probability and White Noise Analysis, Proceedings of the 29th Conference in Hammamet, Tunisia, 1318 October 2008. H. Ouerdiane and A Barhoumi (eds.), World Scientific, **25**, (2010) 153–184) is the study of the robustness of the sensitivity with respect to parameters in expectation functionals with respect to various approximations of a Lévy process. Chapter 5 (extracted from the article "Robustness of option prices and their deltas in markets modeled by jump-diffusions" by Fred Espen Benth, Giulia Di Nunno, and Asma Khedher, available as *E-print* No. 2, January (2010), Department of Mathematics, University of Oslo, Norway, to appear in *Comm. Stoch. Analysis*) is the study of the problem of robustness of the delta to model choice for options written in jump-diffusion models. In Chapter 6 (extracted from the article "Computation of Greeks in multi-factor models with applications to power and commodity markets" by Fred Espen Benth, Giulia Di Nunno, and Asma Khedher, available as *E-print*, No. 5, March (2010), Department of Mathematics, University of Oslo, Norway, submitted for publication) we apply the conditional density method for the computation of the Greeks written in multi-factor dynamics and we apply this to power and commodity markets. In Chapter 7 (extracted from the article "Computation of the delta in multidimensional jump-diffusion setting with applications to stochastic volatility models" by Asma Khedher, available as *E-print*, April (2011), Department of Mathematics, University of Oslo, Norway, submitted for publication) we apply the computation of the delta to stochastic volatility models.

Though not in a strict chronological order, we choose to present the paper in the above order to establish a coherent and consistent exposition of the material in the thesis.

Mathematical preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration \mathcal{F}_t , $t \in [0, T]$, ($T > 0$) satisfying the *usual conditions* (see Karatzas and Shreve [49]). We introduce the generic notation $L = L(t)$, $0 \leq t \leq T$, for a Lévy process on the given probability space and denote by $B = B(t)$, $0 \leq t \leq T$, a Brownian motion independent of L . we set $L(0) = B(0) = 0$ and we work with the RCLL¹ version of the Lévy process, using the notation $\Delta L(t) := L(t) - L(t-)$. Denote the Lévy measure of L by $\ell(dz)$. Recall that $\ell(dz)$, $z \in \mathbb{R}_0$, is a σ -finite Borel measure on $\mathbb{R}_0 := \mathbb{R} - \{0\}$.

We also recall the Lévy-Itô decomposition of a Lévy process (see Sato [59]):

Theorem 2.0.1. *For $t \geq 0$, let L be a Lévy process on \mathbb{R} and ℓ its Lévy measure. Then we have:*

- ℓ verifies

$$\int_{\mathbb{R}_0} \min(1, z^2) \ell(dz) < \infty.$$

- The jump measure of L , denoted by $N(dt, dz)$, is a Poisson random measure on $[0, \infty) \times \mathbb{R}_0$ with intensity measure $\ell(dz) dt$.
- There exists a Brownian motion $W(t)$, $0 \leq t \leq T$ and two constants $a, b \in \mathbb{R}$ such that

$$L(t) = at + bW(t) + Z(t) + \lim_{\varepsilon \downarrow 0} \tilde{Z}_\varepsilon(t), \quad (2.1)$$

where

$$Z(t) := \sum_{s \in [0, t]} \Delta L(s) \mathbf{1}_{\{|\Delta L(s)| \geq 1\}} = \int_0^t \int_{|z| \geq 1} z N(ds, dz)$$

and

$$\tilde{Z}_\varepsilon(t) := \sum_{s \in [0, t]} \Delta L(s) \mathbf{1}_{\{\varepsilon \leq |\Delta L(s)| < 1\}} - t \int_{\varepsilon \leq |z| < 1} z \ell(dz) = \int_0^t \int_{\varepsilon \leq |z| < 1} z \tilde{N}(ds, dz),$$

¹Right-continuous with left limits, also called *càdlàg*.

where \tilde{N} is the compensated Poisson random measure of L . The convergence of $\tilde{Z}_\varepsilon(t)$ in (2.1) is almost sure and uniform on $t \in [0, T]$. The components W , Z and \tilde{Z}_ε are independent.

In various applications involving statistical and numerical methods, it is often useful to approximate the small jumps by a scaled Brownian motion. This approximation was advocated in Rydberg [57] as a way to simulate the path of a Lévy process with NIG distributed increments, and later studied in detail by Asmussen and Rosinski [3]. We shall make use of it to study the robustness of option prices and their deltas based on Lévy models and jump-diffusion models (see Chapters 4 and 5).

We introduce the following notation for the variation of the Lévy process L close to the origin. For $0 < \varepsilon \leq 1$, define

$$\sigma^2(\varepsilon) := \int_{|z| < \varepsilon} z^2 \ell(dz), \quad 0 < \varepsilon \leq 1. \quad (2.2)$$

Since every Lévy measure $\ell(dz)$ integrates z^2 in an open interval around zero, we have that $\sigma^2(\varepsilon)$ is finite for any $\varepsilon > 0$. Note that the $\sigma^2(\varepsilon)$ is the variance of the jumps smaller than ε of L in the case it is symmetric and has mean zero. By dominated convergence $\sigma^2(\varepsilon)$ converges to zero when $\varepsilon \downarrow 0$.

Recall the Lévy-Itô decomposition of a Lévy process L and introduce now an approximating Lévy process (in law)

$$L_\varepsilon(t) := at + bW(t) + \sigma(\varepsilon)B(t) + Z(t) + \tilde{Z}_\varepsilon(t), \quad (2.3)$$

with $\sigma^2(\varepsilon)$ as in (2.2), and B being a Brownian motion independent of L (which in particular means independent of W). From the definition of \tilde{Z}_ε , we see that we have substituted the small jumps (compensated by their expectation) in L by a Brownian motion scaled with $\sigma(\varepsilon)$, the standard deviation of the compensated small jumps. We have the following result

Proposition 2.0.1. *Let the process L respectively L_ε be defined as in equation (2.1), respectively (2.3). Then, for every $t \geq 0$,*

$$\lim_{\varepsilon \rightarrow 0} L_\varepsilon(t) = L(t) \quad \mathbb{P} - a.s.$$

In fact, the limit above also holds in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E}[|L_\varepsilon(t) - L(t)|] \leq 2\sigma(\varepsilon)\sqrt{t}.$$

Proof. The \mathbb{P} -a.s. convergence follows from the proof of the Lévy-Kintchine formula (See Thm. 19.2 in Sato [59]). Concerning the L^1 -convergence, we argue as follows. The combined application of the triangle and Cauchy-Schwarz inequalities gives

$$\begin{aligned} \mathbb{E}[|L_\varepsilon(t) - L(t)|] &= \mathbb{E}\left[|\sigma(\varepsilon)B(t) - \int_0^t \int_{0 < |z| < \varepsilon} z \tilde{N}(ds, dz)|\right] \\ &\leq \sigma(\varepsilon)\mathbb{E}[|B(t)|] + \mathbb{E}\left[\left|\int_0^t \int_{0 < |z| < \varepsilon} z \tilde{N}(ds, dz)\right|\right] \end{aligned}$$

$$\begin{aligned}
&\leq \sigma(\varepsilon) \mathbb{E} [B^2(t)]^{1/2} + \mathbb{E} \left[\left(\int_0^t \int_{0 < |z| < \varepsilon} z \tilde{N}(ds, dz) \right)^2 \right]^{1/2} \\
&\leq 2\sigma(\varepsilon) \sqrt{t}.
\end{aligned}$$

This proves the proposition. \square

We shall make use of the approximation and its convergence properties in our analysis. The study in Asmussen and Rosinski [3] gives a central limit type of result for the approximation of the small jumps. It says that the small jumps, after scaling by $\sigma(\varepsilon)$, are indeed close to be standard normally distributed. We note that the above result only says that, for every t , the two random variables L and L_ε are close in distribution, but nothing about the asymptotic distribution of the small jumps in the limit. Indeed, under an asymptotic condition on $\sigma(\varepsilon)$, the result in [3] is:

Theorem 2.0.2. *If*

$$\lim_{\varepsilon \rightarrow 0} \frac{\sigma(\varepsilon)}{\varepsilon} = \infty, \quad (2.4)$$

then

$$\lim_{\varepsilon \rightarrow 0} \sigma^{-1}(\varepsilon) \tilde{Z}_\varepsilon = B,$$

where B is a Brownian motion and the convergence is in distribution.

This result supports the choice of using a Brownian motion and the scale $\sigma(\varepsilon)$ for the small jumps of a Lévy process. We will frequently make use of $\sigma(\varepsilon)$ for our studies. But first, we recall a result of Orey [55] which relates the asymptotic behavior of the Lévy measure at zero (that is, the asymptotic behavior of $\sigma^2(\varepsilon)$ as ε tends to zero) to the smoothness of the probability density of L .

Theorem 2.0.3. *Let L be a Lévy process, then it follows:*

- *If $b > 0$ or $\ell(\mathbb{R}_0) = \infty$, then L has a continuous probability density $p_t(\cdot)$ on \mathbb{R} .*
- *If there exists $\gamma \in]0, 2[$ such that $\ell(dz)$ satisfies*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\sigma^2(\varepsilon)}{\varepsilon^\gamma} > 0, \quad (2.5)$$

then the probability density p_t of L is infinitely continuously differentiable and for all $n \geq 1$,

$$\lim_{|x| \rightarrow \infty} \frac{\partial^n p_t}{\partial x^n}(x) = 0.$$

We observe that both the α -stable and the normal inverse Gaussian (NIG) Lévy processes satisfy condition (2.5) ensuring the existence of a smooth density. Indeed, the Lévy measure of an α -stable process with $\alpha \in]0, 2[$ is (see for instance Sato [59])

$$\ell(dz) = c_1 |z|^{-1-\alpha} \mathbf{1}_{\{z < 0\}} dz + c_2 z^{-1-\alpha} \mathbf{1}_{\{z > 0\}} dz,$$

with $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Therefore,

$$\sigma^2(\varepsilon) = \frac{c_1 + c_2}{2 - \alpha} \varepsilon^{2-\alpha}.$$

Hence, choose $\gamma = 2 - \alpha$ to verify condition (2.5). The NIG Lévy process has Lévy measure (see Barndorff-Nielsen [14]),

$$\ell(dz) = \frac{\alpha\delta}{\pi|z|} K_1(\alpha|z|) e^{\beta z} dz,$$

where α, β, δ are parameters satisfying $0 \leq \beta \leq \alpha$ and $\delta > 0$, and $K_1(z)$ is the modified Bessel function of the third kind with index 1. Using properties of the Bessel functions (see Asmussen and Rosinski [3]), one finds

$$\sigma^2(\varepsilon) = \frac{2\delta}{\pi} \varepsilon.$$

Hence, letting $\gamma = 1$ we readily verify condition (2.5) also for the NIG Lévy process.

Thm. 2.0.3 is useful in our analysis since it ensures that the density function of a Lévy process is differentiable, which is the basic requirement for the applicability of the so-called density method which we study in Chapter 4.

We are concerned with the derivative of the expectation of functionals of the form

$$F(x) := \mathbb{E}[f(x + Y)], \quad (2.6)$$

for a random variable Y and a measurable function f such that $f(x + Y) \in L^1(\mathbb{P})$ for each $x \in \mathbb{R}$ (or in some subset of \mathbb{R}). Here, we denote by $L^1(\mathbb{P})$ the space of all random variables which are integrable with respect to \mathbb{P} . In most of our forthcoming analysis, Y will be a Lévy process L or a jump-diffusion X or some approximation of such. We call a random variable π a *weight* if $f(x + Y)\pi \in L^1(\mathbb{P})$ for $x \in \mathbb{R}$ and

$$F'(x) := \frac{dF(x)}{dx} = \mathbb{E}[f(x + Y)\pi]. \quad (2.7)$$

A straightforward derivation inside the expectation operator would lead to $F'(x) = \mathbb{E}[f'(x + Y)]$, so a sensitivity weight can be viewed as the result after a kind of “integration-by-parts” operation. The advantage of an expression of the form (2.7) is that we can consider the derivative of expectation functionals where the function f is not differentiable. Examples where this is relevant include the calculation of delta-hedge ratios in option pricing for “payoff-functions” f being non-differentiable (digital options, say). Other examples are the sensitivity of risk measures with respect to a parameter, where the risk measure may be a non-differentiable function of the risk (Value-at-Risk, say, which is a quantile measure).

There exist by now at least two methods to derive sensitivity weights for functionals like $F(x)$. The classical approach is the density method, which transfers the dependency of x to the density function of Y , and then differentiate. An alternative method is the Malliavin approach, applying the tools from Malliavin calculus to perform an integration-by-parts utilizing the Malliavin derivative rather than classical differentiation. We refer to Fournié *et al.* [34] for more information on this approach.

2.1 The density method

Let us discuss the density method (see Broadie and Glasserman [13] for applications to finance). Suppose Y has a density p_Y with respect to the Lebesgue measure dt . Then, from classical probability theory, we have that

$$F(x) = \int_{\mathbb{R}} f(x+y)p_Y(y) dy = \int_{\mathbb{R}} f(y)p_Y(y-x) dy. \quad (2.8)$$

Hence, the expectation functional $F(x)$ can be expressed as a convolution between f and p_Y . Recalling Thm. 8.10 in Folland [33], as long as $f \in L^1(\mathbb{R})$ and $p_Y \in C_0^n(\mathbb{R})$, F is n times continuously differentiable and its derivatives can be expressed as

$$F^{(k)}(x) = \int_{\mathbb{R}} f(y)(-1)^k \frac{d^k}{dy^k} p_Y(y-x) dy,$$

for $k \leq n$ and $F^{(k)}$ denoting the k 'th derivative of F . Here we have denoted the space of Lebesgue integrable functions on \mathbb{R} by $L^1(\mathbb{R})$ and the space of differentiable (up to order n) functions on \mathbb{R} vanishing at infinity by $C_0^n(\mathbb{R})$.

Restricting our attention to $n = 1$, and assuming that $p_Y(y) > 0$ for $y \in \mathbb{R}$, we find that

$$F'(x) = \int_{\mathbb{R}} f(x+y) \left(-\frac{d}{dy} \ln p_Y(y)\right) p_Y(y) dy = \mathbb{E}[f(x+Y)(-\partial \ln p_Y(Y))].$$

Thus, the density method yields a weight $\pi = -\partial \ln p_Y(Y)$, the logarithmic derivative of the density. As we see from above, under very mild assumptions on the density of Y and the function f , we can find a weight π for calculating the derivative of F without having to differentiate f .

Assuming that $f \in L^1(\mathbb{R})$ is rather strict in many applications. We can relax the conditions on f considerably as follows. Suppose that p_Y is differentiable and strictly positive, and $f(\cdot)p_Y'(\cdot - x)$ is bounded uniformly in x by an integrable function on \mathbb{R} . Then, according to Thm. 2.27 in Folland [33], we have

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_{\mathbb{R}} f(y)p_Y(y-x) dy \\ &= \int_{\mathbb{R}} f(y)(-1)p_Y'(y-x) dy \\ &= \int_{\mathbb{R}} f(x+y) \left(-\frac{d}{dy} \ln p_Y(y)\right) p_Y(y) dy \\ &= \mathbb{E}[f(x+Y)(-\partial \ln p_Y(Y))]. \end{aligned} \quad (2.9)$$

We obtain the same weight $\pi = -\partial \ln p_Y(Y)$ as above, naturally. However, we can include functions f which can grow at infinity as long as the density (and its derivative) dampens this growth sufficiently. This ensures that we can apply the density method in financial contexts like calculating the delta of a call option.

2.2 The Malliavin method

In this Section, we review the method of Fournié et. al [34] to derive the stochastic weight π for calculating the derivative $F'(x)$. Let $Dom D^W$ be the set of Malliavin differentiable random variables for Gaussian processes and D^W the Malliavin operator (see Nualart [54] for the Malliavin derivative in the Wiener space). We consider the case when the underlying price process is a Markov diffusion $Y \in Dom D^W$ of the form

$$\begin{cases} dY(t) = \mu(Y(t))dt + \sigma(Y(t))dW(t), \\ Y(0) = x, \quad x > 0. \end{cases} \quad (2.10)$$

Assume that μ and σ are continuously differentiable functions with bounded derivatives. We associate with the process Y , a process V given by:

$$\begin{cases} dV(t) = \mu'(Y(t))V(t)dt + \sigma'(Y(t))V(t)dW(t), \\ V(0) = 1. \end{cases} \quad (2.11)$$

The process V is called the first variation process for Y and we have

$$V(t) = \frac{\partial Y(t)}{\partial x}.$$

Proposition 2.2.1. [34] *Let Y be a process of the form (2.10). Then for all $t \geq 0$,*

$$D_s^W Y(t) = V(t)V(s)^{-1}\sigma(Y(s))\mathbf{1}_{\{s \leq t\}}, \quad s \geq 0.$$

Proof. We have

$$Y(t) = x + \int_0^t \mu(Y(u))du + \int_0^t \sigma(Y(u))dW(u).$$

Thus the derivative of Y at time s is given by

$$\begin{aligned} D_s^W Y(t) &= D_s^W \left(\int_0^t \mu(Y(u))du \right) + D_s^W \left(\int_0^t \sigma(Y(u))dW(u) \right) \\ &= \int_s^t D_s^W \left(\mu(Y(u)) \right) du + \int_s^t D_s^W \left(\sigma(Y(u)) \right) dW(u) + \sigma(Y(s)) \\ &= \int_s^t \mu'(Y(u))D_s^W Y(u)du + \int_s^t \sigma'(Y(u))D_s^W Y(u)dW(u) + \sigma(Y(s)). \end{aligned}$$

Take $Z(t) = D_s^W Y(t)$, this represents the equation of the derivative of $Y(t)$ at time s fixed. For $t \geq s$,

$$\begin{cases} dZ(t) = \mu'(Y(t))Z(t)dt + \sigma'(Y(t))Z(t)dW(t), \\ Z(s) = \sigma(Y(s)). \end{cases}$$

The processes Z and V verify the same differential equations with different initial conditions, therefore

$$Z(t) = \lambda V(t)\mathbf{1}_{\{s \leq t\}}, \quad t \geq s,$$

where $\lambda = \sigma(Y(s))V(s)^{-1}$. Then

$$D_s^W Y(t) = V(t)V(s)^{-1}\sigma(Y(s))\mathbf{1}_{\{s \leq t\}}.$$

□

Proposition 2.2.2. [34] *Let $f(Y(T)) \in L^2(\Omega)$ and Y be a process of the form (2.10). Define*

$$\Gamma = \left\{ a \in L^2[0, T] \mid \int_0^T a(t) dt = 1 \right\}$$

and

$$\pi = \int_0^T a(t) V(t) \sigma^{-1}(Y(t)) dW(t).$$

If $a \in \Gamma$ and $(\mathbb{E}[\pi^2])^{1/2} < \infty$, then

$$F'(x) = \mathbb{E} \left[f(Y(T)) \pi \right].$$

Proof. First, assume that $f \in C_K^\infty(\mathbb{R})$, the set of infinitely differentiable functions with compact support, then

$$\begin{aligned} F'(x) &= \frac{\partial}{\partial x} \mathbb{E} \left[f(Y(T)) \right] = \mathbb{E} \left[\frac{\partial}{\partial x} f(Y(T)) \right] = \mathbb{E} \left[f'(Y(T)) \frac{\partial Y(T)}{\partial x} \right] \\ &= \mathbb{E} \left[f'(Y(T)) V(T) \right], \end{aligned}$$

where V is the first variation process of Y . We want to write the expression $\mathbb{E} \left[f'(Y(T)) V(T) \right]$ as $\mathbb{E} \left[f(Y(T)) \delta(\eta) \right]$, where $\delta(\eta)$ is the Skorohod integral with respect to the Brownian motion W of a certain $\eta \in L^2(\Omega \times [0, T])$. By the integration by parts formula, we have

$$\begin{aligned} \mathbb{E} \left[f(Y(T)) \delta(\eta) \right] &= \mathbb{E} \left[\int_0^T D_s^W(f(Y(T)) \eta(s)) ds \right] \\ &= \mathbb{E} \left[\int_0^T f'(Y(T)) D_s^W(Y(T)) \eta(s) ds \right] \\ &= \mathbb{E} \left[f'(Y(T)) \int_0^T V(T) (V(s))^{-1} \sigma(Y(s)) \mathbf{1}_{\{s \leq T\}} \eta(s) ds \right]. \end{aligned}$$

Therefore η should verify the following equation

$$V(T) = \int_0^T V(T) (V(s))^{-1} \sigma(Y(s)) \mathbf{1}_{\{s \leq T\}} \eta(s) ds. \quad (2.12)$$

For $a \in \Gamma$, we have

$$\eta(t) = a(t) V(t) \sigma(Y(t))^{-1}. \quad (2.13)$$

Therefore

$$F'(x) = \mathbb{E} \left[f(Y(T)) \int_0^T a(t) V(t) \sigma^{-1}(Y(t)) dW(t) \right].$$

Now, let $f(Y(T)) \in L^2(\Omega)$. Then $f(x) \in L^2(\mathbb{R}, p_{Y(T)})$, where $p_{Y(T)}$ is the probability density of $Y(T)$. Therefore

$$\exists (f_n)_{n \in \mathbb{N}} \in C_K^\infty(\mathbb{R}) \text{ such that } \lim_{n \rightarrow \infty} f_n = f, \quad \text{the limit is in } L^2(\mathbb{R}, p_{Y(T)}).$$

We denote by

$$u(x) = \mathbb{E}[f(Y(T))] \quad \text{and} \quad u_n(x) = \mathbb{E}[f_n(Y(T))].$$

As the convergence in L^2 implies the convergence in L^1 , $(u_n)_{n \in \mathbb{N}}$ converges point wise to u and for $x \in \mathbb{R}$, we have

$$\lim_{n \rightarrow \infty} u_n(x) = u(x).$$

As $f_n \in C_K^\infty(\mathbb{R})$, then

$$\frac{\partial}{\partial x} \mathbb{E}[f_n(Y(T))] = \mathbb{E}[f_n(Y(T))\pi].$$

We denote by $g(x) = \mathbb{E}[f(Y(T))\pi]$. By Cauchy-Schwartz inequality, we have

$$|g(x) - \frac{\partial}{\partial x} u_n(x)| = |\mathbb{E}[(f(Y(T)) - f_n(Y(T)))\pi]| \leq \left(\mathbb{E}[\pi^2] \right)^{1/2} \psi_n(x). \quad (2.14)$$

where $\psi_n(x) = \left(\mathbb{E}[\{f(Y(T)) - f_n(Y(T))\}^2] \right)^{1/2}$. The convergence of u_n implies the convergence of ψ_n to 0 point wise when n tends to infinity. Therefore the sequence $(\frac{\partial}{\partial x} u_n(x))_{n \in \mathbb{N}}$ converges point wise to $g(x)$. As the function $\left(\mathbb{E}[\pi^2] \right)^{1/2}$ is finite, then the equation (2.14) shows that the convergence is uniform in every compact $K \in \mathbb{R}$. Therefore the function u is differentiable and its derivative is equal to g and the result holds for $f(Y(T)) \in L^2(\Omega)$. \square

A note on convergence of option prices and their Greeks for Lévy models

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In this chapter, we study the robustness of option prices to model variation after a change of measure where the measure depends on the model choice. We consider geometric Lévy models in which the infinite activity of the small jumps is approximated by a scaled Brownian motion. For the Esscher transform, the minimal entropy martingale measure, the minimal martingale measure and the mean variance martingale measure, we show that the option prices and their corresponding deltas converge as the scaling of the Brownian motion part tends to zero. We give some examples illustrating our results.

The chapter is organized as follows. In Section 3.1, we present the stock price model which is a geometric Lévy process. In Section 3.2, we show the stability of option prices after a change of measure.

3.1 Framework: two models for the stock price dynamics

Recall from (2.1) the Lévy process L . Let $S = S(t)$, $0 \leq t \leq T$, be a geometric Lévy process defined by

$$S(t) = S(0)e^{L(t)}, \quad S(0) > 0.$$

This represents a given stock price under the physical measure \mathbb{P} . We consider the discounted stock price process $\hat{S} = \hat{S}(t)$, $0 \leq t \leq T$, given by

$$\hat{S}(t) = e^{-rt}S(t), \quad \hat{S}(0) = S(0)$$

where the constant $r > 0$ is the risk-free instantaneous interest rate. Assuming exponential integrability of the Lévy measure,

$$\int_1^\infty e^z \ell(dz) < \infty,$$

we apply the Itô formula, to represent the process S as the solution of the following linear stochastic differential equation (SDE)

$$S(t) = S(0) + \int_0^t S(s-) d\widehat{L}(s),$$

where

$$\widehat{L}(t) = a_1 t + bW(t) + \int_0^t \int_{\mathbb{R}_0} (e^z - 1) \widetilde{N}(ds, dz). \quad (3.1)$$

Here

$$a_1 = a + \frac{1}{2}b^2 + \int_{\mathbb{R}_0} \{e^z - 1 - z\mathbf{1}_{|z| \leq 1}\} \ell(dz).$$

Using the Itô formula again, we can represent the discounted stock price \widehat{S} as the solution of the following linear SDE

$$d\widehat{S}(t) = (a_1 - r)\widehat{S}(t-)dt + b\widehat{S}(t-)dW(t) + \widehat{S}(t-) \int_{\mathbb{R}_0} (e^z - 1) \widetilde{N}(dt, dz). \quad (3.2)$$

These representations will be useful in our later considerations.

The second stock price dynamics $S_\varepsilon = S_\varepsilon(t)$, $0 \leq t \leq T$, are given by

$$S_\varepsilon(t) = S(0)e^{L_\varepsilon(t)}, \quad S(0) > 0, \quad (3.3)$$

with L_ε defined in (2.3). Thus, we have taken the dynamics $S(t)$ and substituted the small jumps of L with a Brownian motion appropriately scaled. We note that by Prop. 2.0.1, $S_\varepsilon(t)$ converges $\mathbb{P} - a.s.$ to $S(t)$, for every t .

As we aim at studying the stability of option prices under a change of measure, we need to introduce the notion of (local) martingale measures for the discounted price process \widehat{S} . For this purpose, let $\mathcal{P}(\Omega, \mathcal{F})$ be the set of all probability measures on (Ω, \mathcal{F}) . We introduce some sets of probability measures on (Ω, \mathcal{F}_T) . First, $ACLLM(\mathcal{P})$ is the set of *absolutely continuous local martingale measures*,

$$ACLLM(\mathcal{P}) := \{\widehat{\mathbb{P}} \in \mathcal{P}(\Omega, \mathcal{F}) : \widehat{\mathbb{P}} \ll \mathbb{P} \text{ on } \mathcal{F}_T \text{ and } \widehat{S} \text{ is a local martingale under } \widehat{\mathbb{P}}\}.$$

Next, $EMM(\mathcal{P})$ is the set of *equivalent martingale measures* for \widehat{S} ,

$$EMM(\mathcal{P}) := \{\widehat{\mathbb{P}} \in \mathcal{P}(\Omega, \mathcal{F}) : \widehat{\mathbb{P}} \sim \mathbb{P} \text{ on } \mathcal{F}_T \text{ and } \widehat{S} \text{ is a martingale under } \widehat{\mathbb{P}}\}.$$

We may introduce sets for \widehat{S}_ε analogously.

The following theorem, due to Tankov [62], states the conditions for the absence of arbitrage in exponential Lévy models.

Theorem 3.1.1. *Let L be a Lévy process as defined in (2.1). The following statements are equivalent*

1. *There exists a probability $\widetilde{\mathbb{P}}$ equivalent to \mathbb{P} such that L is a Lévy process under $\widetilde{\mathbb{P}}$ and e^L is a martingale.*

2. Either $L = 0$ or L is not \mathbb{P} -a.s. monotone.

3. One of the following conditions is satisfied:

- $b > 0$.
- $b = 0$ and $\int_{|x| \leq 1} |x| \ell(dx) = \infty$.
- $b = 0$, $\int_{|x| \leq 1} |x| \ell(dx) < \infty$, $\ell((-\infty, 0)) > 0$, and $\ell((0, \infty)) > 0$.
- $b = 0$, $\int_{|x| \leq 1} |x| \ell(dx) < \infty$, $\ell((-\infty, 0)) > 0$, and $a - \int_{|x| \leq 1} x \ell(dx) > 0$.
- $b = 0$, $\int_{|x| \leq 1} |x| \ell(dx) < \infty$, $\ell((0, \infty)) > 0$, and $a - \int_{|x| \leq 1} x \ell(dx) < 0$.

In the following, we assume that our models do not allow for arbitrage.

3.2 Stability of option prices under a change of measure

In this section we study the convergence of prices of options written on S_ε to the corresponding prices written on S . We consider different choices of equivalent martingale measures widely used in the financial literature. Note that the measures themselves depend on the approximating stock price dynamics.

3.2.1 The Esscher transform

The moment generating function of $L(t)$, for any t , is given by

$$\begin{aligned} M_t(\theta) &= \mathbb{E}[e^{\theta L(t)}] \\ &= \exp\left\{t\left(a\theta + \frac{1}{2}b^2\theta^2 + \int_{\mathbb{R}_0} (e^{\theta z} - 1 - z\mathbf{1}_{|z| < 1}\theta)\ell(dz)\right)\right\}, \quad |\theta| < M, \end{aligned} \quad (3.4)$$

for some $0 < M \leq \infty$ for which we have

$$\int_{|z| > 1} e^{\theta z} \ell(dz) < \infty, \quad |\theta| < M, \quad (3.5)$$

see Theorem 25.17 in Sato [59]. Set

$$G(\theta) := a\theta + \frac{1}{2}b^2\theta^2 + \int_{\mathbb{R}_0} (e^{\theta z} - 1 - z\mathbf{1}_{|z| < 1}\theta)\ell(dz).$$

The Esscher transform is defined as a probability measure $\tilde{\mathbb{P}}_\theta \sim \mathbb{P}$ (see Gerber and Shiu [42]) such that

$$\begin{aligned} \frac{d\tilde{\mathbb{P}}_\theta}{d\mathbb{P}} \Big|_{\mathcal{F}_t} &= \exp(\theta L(t) - tG(\theta)) \\ &= \exp\left\{\theta bW(t) - \frac{1}{2}b^2\theta^2 t + \theta \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz) - t \int_{\mathbb{R}_0} (e^{\theta z} - 1 - z\theta)\ell(dz)\right\}. \end{aligned}$$

We denote by $\tilde{\mathbb{E}}_\theta$ the expectation under the new measure $\tilde{\mathbb{P}}_\theta$.

In applications to finance, the risk neutral Esscher measure is defined as the $\tilde{\mathbb{P}}_\theta$ such that the process $\hat{S}(t) = e^{-rt}S(t)$, $0 \leq t \leq T$, is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. The condition

$$\tilde{\mathbb{E}}_\theta[e^{-rt}S(t)] = S(0)$$

yields

$$\mathbb{E}[e^{L(t)}e^{\theta L(t)-tG(\theta)}] = e^{rt}$$

which is equivalent to

$$G(\theta + 1) - G(\theta) = r. \quad (3.6)$$

Condition (3.6) is necessary and sufficient for $\tilde{\mathbb{P}}_\theta \in EMM(\mathcal{P})$. From the definition of $G(\theta)$, we see that (3.6) becomes

$$\begin{aligned} a(1 + \theta) + \frac{1}{2}(1 + \theta)^2 b^2 + \int_{\mathbb{R}_0} \{e^{(\theta+1)z} - 1 - z\mathbf{1}_{|z| < 1}(\theta + 1)\} \ell(dz) \\ - a\theta - \frac{1}{2}\theta^2 b^2 - \int_{\mathbb{R}_0} \{e^{\theta z} - 1 - z\mathbf{1}_{|z| < 1}\theta\} \ell(dz) = r. \end{aligned}$$

Hence

$$\begin{aligned} a - r + b^2\theta + \frac{1}{2}b^2 + \int_{\mathbb{R}_0} e^{\theta z}(e^z - 1 - z\mathbf{1}_{|z| < 1})\ell(dz) \\ + \int_{0 < |z| < 1} z(e^{\theta z} - 1)\ell(dz) = 0. \end{aligned}$$

Define

$$g(\theta) := b^2\theta + \int_{|z| \geq 0} e^{\theta z}(e^z - 1 - z\mathbf{1}_{|z| < 1})\ell(dz) + \int_{0 \leq |z| < 1} z(e^{\theta z} - 1)\ell(dz).$$

Under the arbitrage conditions, Gerber and Shiu [43] proved that equation (3.6) admits a unique solution in \mathbb{R} if and only if one of these two conditions is fulfilled

- $M = \infty$,
- $M < \infty$ and $r - a - \frac{1}{2}b^2 \in \left(\lim_{\theta \rightarrow -M} g(\theta), \lim_{\theta \rightarrow M} g(\theta) \right]$.

The stochastic process L is still a Lévy process under the probability measure $\tilde{\mathbb{P}}_\theta$. In this sense we say that the Esscher transform is structure preserving, see Theorem 33.1 in Sato [59]. The new characteristic triplet of L under $\tilde{\mathbb{P}}_\theta$ is given by $(b^2, \tilde{\ell}, \tilde{a})$, where

$$\tilde{\ell}(dz) = e^{\theta z}\ell(dz)$$

and

$$\tilde{a} = a + b^2\theta + \int_{|z| < 1} z(e^{\theta z} - 1)\ell(dz). \quad (3.7)$$

Next, we consider the approximating price process $S_\varepsilon(t)$ and its discounted version $\widehat{S}_\varepsilon(t) = e^{-rt}S_\varepsilon(t)$, $0 \leq t \leq T$. We define

$$G_\varepsilon(\theta) := a\theta + \frac{1}{2}\theta^2(b^2 + \sigma^2(\varepsilon)) + \int_{|z| \geq \varepsilon} (e^{\theta z} - 1 - z\mathbf{1}_{|z| < 1}\theta)\ell(dz).$$

Note that for $G_\varepsilon(\theta)$ to exist, the Condition (3.5) is still sufficient. An Esscher probability measure $\widetilde{\mathbb{P}}_\theta^\varepsilon \sim \mathbb{P}$ is given by

$$\begin{aligned} \left. \frac{d\widetilde{\mathbb{P}}_\theta^\varepsilon}{d\mathbb{P}} \right|_{\mathcal{F}_t} &= \exp(\theta L_\varepsilon(t) - tG_\varepsilon(\theta)) \\ &= \exp\left\{\theta(bW(t) + \sigma(\varepsilon)B(t)) - \frac{1}{2}(b^2 + \sigma^2(\varepsilon))\theta^2 t + \theta \int_0^t \int_{|z| \geq \varepsilon} z\widetilde{N}(ds, dz) \right. \\ &\quad \left. - t \int_{|z| \geq \varepsilon} (e^{\theta z} - 1 - z\theta)\ell(dz)\right\}. \end{aligned} \quad (3.8)$$

By the same argument as above, we can see that $\widetilde{\mathbb{P}}_\theta^\varepsilon$ is a risk-neutral equivalent martingale measure if and only if the parameter θ satisfies

$$G_\varepsilon(\theta + 1) - G_\varepsilon(\theta) = r. \quad (3.9)$$

As in Gerber and Shiu [43], one can prove the existence and uniqueness of the parameter θ_ε solving (3.9), for ε fixed in $(0, 1)$. We adapt their proof to our model.

Lemma 3.2.1. *Define*

$$g_\varepsilon(\theta) := (b^2 + \sigma^2(\varepsilon))\theta + \int_{|z| \geq \varepsilon} e^{\theta z}(e^z - 1 - z\mathbf{1}_{|z| < 1})\ell(dz) + \int_{\varepsilon \leq |z| < 1} z(e^{\theta z} - 1)\ell(dz).$$

Then, for each $\varepsilon \in (0, 1)$ the solution of

$$G_\varepsilon(1 + \theta) - G_\varepsilon(\theta) = r$$

exists and is unique in \mathbb{R} if and only if one of the following two conditions is satisfied

$$M = \infty, \quad (3.10)$$

$$M < \infty \quad \text{and} \quad r - a - \frac{1}{2}(\sigma^2(\varepsilon) + b^2) \in \left(\lim_{\theta \rightarrow -M} g_\varepsilon(\theta), \lim_{\theta \rightarrow M} g_\varepsilon(\theta) \right]. \quad (3.11)$$

We denote this solution θ_ε emphasizing the dependence on $\varepsilon \in (0, 1)$.

Proof. By dominated convergence for ε fixed, the function $g_\varepsilon(\theta)$ is differentiable with derivative given by

$$g'_\varepsilon(\theta) = (b^2 + \sigma^2(\varepsilon)) + \int_{|z| \geq \varepsilon} z(e^z - 1)e^{\theta z}\ell(dz).$$

Note that $z(\exp(z) - 1) > 0$ when $|z| \geq \varepsilon$. Hence, since $g'_\varepsilon(\theta) \geq \sigma^2(\varepsilon) > 0$, it follows that $g_\varepsilon(\theta)$ is a strictly increasing function. Moreover, $g_\varepsilon(+\infty) = +\infty$ and $g_\varepsilon(-\infty) = -\infty$. Therefore, the equation $g_\varepsilon(\theta) + a - r + \frac{1}{2}(b^2 + \sigma^2(\varepsilon)) = 0$ admits a unique solution if and only if one of the conditions (3.10) or (3.11) is satisfied. \square

The stochastic process L_ε is still a Lévy process under the probability measure $\tilde{\mathbb{P}}_{\theta_\varepsilon}^\varepsilon$, with characteristic triplet given by $(b^2 + \sigma^2(\varepsilon), \tilde{\ell}^\varepsilon, \tilde{a}^\varepsilon)$, for

$$\tilde{\ell}_\varepsilon(dz) = e^{\theta_\varepsilon z} \ell(dz)$$

and

$$\tilde{a}_\varepsilon = a + (b^2 + \sigma^2(\varepsilon))\theta_\varepsilon + \int_{\varepsilon \leq |z| < 1} z(e^{\theta_\varepsilon z} - 1)\ell(dz). \quad (3.12)$$

In the sequel, we need the following technical lemma in which we study the behavior of θ_ε when ε goes to 0.

Lemma 3.2.2. *Let $\theta_0 \in \mathbb{R}$ be the solution of (3.6). The parameter θ_ε is bounded uniformly in ε , $\varepsilon \in (0, 1)$, and*

$$|\theta_\varepsilon - \theta_0| \leq C_{\theta_0} \sigma^2(\varepsilon),$$

for a positive constant C_{θ_0} depending on θ_0 .

Proof. Recall the definition of $g_\varepsilon(\theta)$ in Lemma 3.2.1. In the proof of Lemma 3.2.1 we showed that $g_\varepsilon(\theta)$ is differentiable. Moreover, it is increasing in θ . Therefore, the inverse $g_\varepsilon^{-1}(\theta)$ exists, it is differentiable and its derivative is given by $(g_\varepsilon^{-1})'(\theta) = \frac{1}{g'_\varepsilon(\theta)}$. In the case when $b > 0$, we have

$$\begin{aligned} g'_\varepsilon(\theta) &= b^2 + \sigma^2(\varepsilon) + \int_{|z| \geq \varepsilon} z e^{\theta z} (e^z - 1) \ell(dz) \\ &\geq b^2. \end{aligned}$$

Hence $(g_\varepsilon^{-1})'(\theta) \leq \frac{1}{b^2}$. By equations (3.6) and (3.9), we know that θ_ε and θ_0 satisfy the following equations

$$g_\varepsilon(\theta_\varepsilon) = r - a - \frac{1}{2}(b^2 + \sigma^2(\varepsilon))$$

and

$$g_\varepsilon(\theta_0) = r - a - \frac{1}{2}b^2 + \sigma^2(\varepsilon)\theta_0 - \int_{|z| < \varepsilon} e^{\theta_0 z} (e^z - 1 - z\mathbf{1}_{|z| < 1}) \ell(dz) - \int_{|z| < \varepsilon} z(e^{\theta_0 z} - 1) \ell(dz),$$

respectively. It follows that

$$\begin{aligned} |\theta_\varepsilon - \theta_0| &= \left| g_\varepsilon^{-1} \left(r - a - \frac{1}{2}(b^2 + \sigma^2(\varepsilon)) \right) - g_\varepsilon^{-1} \left(r - a - \frac{1}{2}b^2 + \sigma^2(\varepsilon)\theta_0 \right) \right. \\ &\quad \left. - \int_{|z| < \varepsilon} e^{\theta_0 z} (e^z - 1 - z\mathbf{1}_{|z| < 1}) \ell(dz) - \int_{|z| < \varepsilon} z(e^{\theta_0 z} - 1) \ell(dz) \right|. \end{aligned}$$

The mean value theorem leads to

$$|\theta_\varepsilon - \theta_0| \leq \frac{1}{b^2} \left| -\frac{1}{2}\sigma^2(\varepsilon) - \sigma^2(\varepsilon)\theta_0 + \int_{|z| < \varepsilon} \{e^{\theta_0 z} (e^z - 1) - z\} \ell(dz) \right|$$

$$\begin{aligned}
 &= \frac{1}{b^2} \left| -\frac{1}{2}\sigma^2(\varepsilon) - \sigma^2(\varepsilon)\theta_0 + \int_{|z|<\varepsilon} \{e^{z(\theta_0+1)} - 1 - z(\theta_0+1)\}\ell(dz) \right. \\
 &\quad \left. - \int_{|z|<\varepsilon} \{e^{z\theta_0} - 1 - z\theta_0\}\ell(dz) \right|.
 \end{aligned}$$

We define $h(\theta) = \int_{|z|<\varepsilon} \{e^{z\theta} - 1 - z\theta\}\ell(dz)$. The function h is differentiable and its derivative is given by $h'(\theta) = \int_{|z|<\varepsilon} z(e^{\theta z} - 1)\ell(dz)$. Thus, applying the mean value theorem to the function $h(\theta)$, $\theta_0 < \theta < \theta_0 + 1$, we get

$$|\theta_\varepsilon - \theta_0| \leq \frac{1}{b^2} \left(\frac{1}{2}\sigma^2(\varepsilon) + \sigma^2(\varepsilon)|\theta_0| + \int_{|z|<\varepsilon} |z|e^{\theta z} - 1|\ell(dz) \right).$$

Applying again the mean value theorem to the function $f(\theta') = e^{\theta' z}$, θ' is an intermediary point between 0 and θ , we get

$$\begin{aligned}
 |\theta_\varepsilon - \theta_0| &\leq \frac{1}{b^2} \left(\frac{1}{2}\sigma^2(\varepsilon) + \sigma^2(\varepsilon)|\theta_0| + \int_{|z|<\varepsilon} |z^2||\theta|e^{\theta' z}\ell(dz) \right) \\
 &\leq \frac{1}{b^2} \left(\frac{1}{2}\sigma^2(\varepsilon) + \sigma^2(\varepsilon)|\theta_0| + e^{|\theta'|}|\theta|\sigma^2(\varepsilon) \right) \\
 &= C_{\theta_0}\sigma^2(\varepsilon).
 \end{aligned}$$

Moreover,

$$|\theta_\varepsilon| \leq M \wedge \left(|\theta_0| + C_{\theta_0}\sigma^2(\varepsilon) \right) \leq |\theta_0| + C_{\theta_0}\sigma^2(1). \quad (3.13)$$

In the case when $b = 0$ and $\theta > \theta_0$, we have $g'_\varepsilon(\theta) \geq \int_{z \geq 1} ze^{\theta_0 z}(e^z - 1)\ell(dz)$ and therefore

$$|\theta_\varepsilon - \theta_0| \leq \frac{1}{\int_{z \geq 1} ze^{\theta_0 z}(e^z - 1)\ell(dz)} \left(\frac{1}{2}\sigma^2(\varepsilon) + \sigma^2(\varepsilon)|\theta_0| + e^{|\theta'|}|\theta|\sigma^2(\varepsilon) \right).$$

When $b = 0$ and $\theta < \theta_0$, we have $g'_\varepsilon(\theta) \geq \int_{z \leq -1} ze^{\theta_0 z}(e^z - 1)\ell(dz)$ and in this case,

$$|\theta_\varepsilon - \theta_0| \leq \frac{1}{\int_{z \leq -1} ze^{\theta_0 z}(e^z - 1)\ell(dz)} \left(\frac{1}{2}\sigma^2(\varepsilon) + \sigma^2(\varepsilon)|\theta_0| + e^{|\theta'|}|\theta|\sigma^2(\varepsilon) \right).$$

Therefore the result also holds in the case when $b = 0$. □

Lemma 3.2.3. *Let $\phi_{L_\varepsilon(T)}$ and $\phi_{L(T)}$ be the characteristic functions of $L_\varepsilon(T)$ under $\tilde{\mathbb{P}}_{\theta_\varepsilon}^\varepsilon$, and $L(T)$ under $\tilde{\mathbb{P}}_\theta$, respectively. Then we have*

$$\lim_{\varepsilon \rightarrow 0} \phi_{L_\varepsilon(T)}(u) = \phi_{L(T)}(u)$$

for every $u \in \mathbb{R}$.

Proof. The characteristic function of $L_\varepsilon(T)$ under $\tilde{\mathbb{P}}_{\theta_\varepsilon}^\varepsilon$ is given by

$$\phi_{L_\varepsilon(T)}(u) = \exp \left\{ i\tilde{a}_\varepsilon T u - \frac{T}{2}(b^2 + \sigma^2(\varepsilon))u + T \int_{|z| \geq \varepsilon} \left(e^{iuz} - 1 - iuz1_{|z|<1} \right) \tilde{\ell}_\varepsilon(dz) \right\}. \quad (3.14)$$

As θ_ε is bounded uniformly in ε , by Prop. 2.24 in Folland [33], we can take the limit inside the integral in equation (3.14) and then the result follows. □

Let us now consider $f \in L^1(\mathbb{R})$, that is, the space of integrable functions on the real line. The Fourier transform of f is defined by

$$\widehat{f}(u) = \int_{\mathbb{R}} f(y) e^{iuy} dy. \quad (3.15)$$

Suppose in addition that $\widehat{f} \in L^1(\mathbb{R})$. Then the inverse Fourier transform is well-defined, and we have

$$f(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iuy} \widehat{f}(u) du. \quad (3.16)$$

With these two definitions at hand, we can do the following calculation taken from Carr and Madan [21] and Eberlein, Glau and Papapantoleon [31]. Assume for every $x \in \mathbb{R}$ that $f(x + \cdot)$ is integrable with respect to the distribution $\widetilde{p}_{L_\varepsilon(T)}(dy)$ of $L_\varepsilon(T)$ under the measure $\widetilde{\mathbb{P}}_{\theta_\varepsilon}$. Then

$$\widetilde{\mathbb{E}}_{\theta_\varepsilon}[f(x + L_\varepsilon(T))] = \int_{\mathbb{R}} f(x + y) \widetilde{p}_{L_\varepsilon(T)}(dy). \quad (3.17)$$

Invoking the representation of f in (3.16), and applying Fubini-Tonelli to commute the integrations, we find

$$\begin{aligned} \widetilde{\mathbb{E}}_{\theta_\varepsilon}[f(x + L_\varepsilon(T))] &= \int_{\mathbb{R}} \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(x+y)u} \widehat{f}(u) du \right\} \widetilde{p}_{L_\varepsilon(T)}(dy) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \left\{ \int_{\mathbb{R}} e^{-iuy} \widetilde{p}_{L_\varepsilon(T)}(dy) \right\} \widehat{f}(u) du. \end{aligned}$$

Thus, it follows that

$$\widetilde{\mathbb{E}}_{\theta_\varepsilon}[f(x + L_\varepsilon(T))] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \phi_{L_\varepsilon(T)}(-u) \widehat{f}(u) du, \quad (3.18)$$

where $\phi_{L_\varepsilon(T)}$ is the characteristic function of $L_\varepsilon(T)$ defined by equation (3.14).

In the setting presented so far, we can conclude the following result which gives the stability of option prices under the Esscher transform.

Proposition 3.2.1. *For $\widehat{f} \in L^1(\mathbb{R})$, we have*

$$\lim_{\varepsilon \rightarrow 0} \widetilde{\mathbb{E}}_{\theta_\varepsilon}[f(x + L_\varepsilon(T))] = \widetilde{\mathbb{E}}_{\theta_0}[f(x + L(T))].$$

In particular, if $\int_{\mathbb{R}} |\widehat{f}(u)|(|u| + |u|^2) du < \infty$, then we have

$$|\widetilde{\mathbb{E}}_{\theta_\varepsilon}[f(x + L_\varepsilon(T))] - \widetilde{\mathbb{E}}_{\theta_0}[f(x + L(T))]| \leq \sigma^2(\varepsilon) C_{\theta_0},$$

where C_{θ_0} is a constant depending on θ_0 .

Proof. From the Fourier representation of the option prices (equation (3.18)), we estimate

$$|\widetilde{\mathbb{E}}_{\theta_\varepsilon}[f(x + L_\varepsilon(T))] - \widetilde{\mathbb{E}}_{\theta_0}[f(x + L(T))]|$$

$$\begin{aligned}
&= \left| \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ e^{-iux} \phi_{L_\varepsilon(T)}(-u) \widehat{f}(u) - e^{-iux} \phi_{L(T)}(-u) \widehat{f}(u) \right\} du \right| \\
&\leq \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(u)| |\phi_{L_\varepsilon(T)}(-u) - \phi_{L(T)}(-u)| du.
\end{aligned}$$

Put

$$x_1 = T \left(\widetilde{a}_\varepsilon u + i \frac{1}{2} (b^2 + \sigma^2(\varepsilon)) u - i \int_{|z| \geq \varepsilon} (e^{iuz} - 1 - iuz \mathbf{1}_{|z| < 1}) \widetilde{\ell}_\varepsilon(dz) \right)$$

and

$$x_2 = T \left(\widetilde{a} u + i \frac{1}{2} b^2 u - i \int_{\mathbb{R}_0} (e^{iuz} - 1 - iuz \mathbf{1}_{|z| < 1}) \widetilde{\ell}(dz) \right).$$

Applying the mean value theorem to the real part of the complex valued function $u(x) = e^{ix}$, we get

$$|\Re(u(x_1) - u(x_2))| \leq K |\Re(x_2 - x_1)|,$$

where K is a positive constant and \Re is the real part of a complex number. Therefore

$$\begin{aligned}
|\Re(u(x_1) - u(x_2))| &\leq KT \left| (\widetilde{a} - \widetilde{a}_\varepsilon) u + \int_{|z| \leq \varepsilon} (\sin(uz) - uz) e^{\theta_0 z} \ell(dz) \right. \\
&\quad \left. + \int_{|z| > \varepsilon} (-\sin(uz) + uz \mathbf{1}_{|z| < 1}) \{e^{\theta_\varepsilon z} - e^{\theta_0 z}\} \ell(dz) \right|.
\end{aligned}$$

From the expressions of \widetilde{a} and $\widetilde{a}_\varepsilon$, in (3.7) and (3.12), respectively, we have

$$\begin{aligned}
|\Re(u(x_1) - u(x_2))| &\leq KT |u| \left(b^2 |\theta_0 - \theta_\varepsilon| + \sigma^2(\varepsilon) |\theta_\varepsilon| + \left| \int_{\varepsilon \leq |z| < 1} z (e^{\theta_0 z} - e^{\theta_\varepsilon z}) \ell(dz) \right| \right. \\
&\quad \left. + \left| \int_{|z| < \varepsilon} z (e^{\theta_0 z} - 1) \ell(dz) \right| \right) \\
&\quad + KT \left(\left| \int_{|z| \leq \varepsilon} (\sin(uz) - uz) e^{\theta_0 z} \ell(dz) \right| \right. \\
&\quad \left. + \left| \int_{|z| > \varepsilon} (-\sin(uz) + uz \mathbf{1}_{|z| < 1}) (e^{\theta_\varepsilon z} - e^{\theta_0 z}) \ell(dz) \right| \right)
\end{aligned}$$

The mean value theorem leads

$$\begin{aligned}
|\Re(u(x_1) - u(x_2))| &\leq KT |u| \left(b^2 |\theta_0 - \theta_\varepsilon| + \sigma^2(\varepsilon) |\theta_\varepsilon| + \int_{\varepsilon \leq |z| < 1} |\theta_0 - \theta_\varepsilon| |z|^2 |e^{\theta_0 z} \ell(dz) \right. \\
&\quad \left. + \left| \int_{|z| < \varepsilon} z (e^{\theta_0 z} - 1) \ell(dz) \right| \right) \\
&\quad + KT \left(\left| \int_{|z| \leq \varepsilon} (\sin(uz) - uz) e^{\theta_0 z} \ell(dz) \right| \right. \\
&\quad \left. + \int_{|z| > \varepsilon} |-\sin(uz) + uz \mathbf{1}_{|z| < 1}| |z| |e^{\theta_0 z} |\theta_0 - \theta_\varepsilon| \ell(dz) \right),
\end{aligned}$$

where θ is an intermediary point between θ_0 and θ_ε . Denote by \Im the imaginary part of a complex number. Applying the mean value theorem to the imaginary part of the complex valued function $u(x) = e^{ix}$ and using the same arguments as above, we get

$$\begin{aligned} |\Im(u(x_1) - u(x_2))| &\leq TK' \left(\frac{1}{2} |u| \sigma^2(\varepsilon) + \left| \int_{|z| \leq \varepsilon} (-\cos(uz) + 1) e^{\theta_0 z} \ell(dz) \right| \right. \\ &\quad \left. + \int_{|z| \geq \varepsilon} |\cos(uz) - 1| |\theta_\varepsilon - \theta_0| |z| e^{\theta z} \ell(dz) \right), \end{aligned}$$

where K' is a positive constant. Therefore

$$\begin{aligned} |\phi_{L_\varepsilon(T)}(-u) - \phi_{L(T)}(-u)| &\leq |\Re(u(x_1) - u(x_2))| + |\Im(u(x_1) - u(x_2))| \\ &\leq CT|u| \left(b^2 C_{\theta_0} \sigma^2(\varepsilon) + \sigma^2(\varepsilon) |\theta_\varepsilon| + \frac{1}{2} \sigma^2(\varepsilon) + \sigma^2(\varepsilon) C_{\theta_0} K_\theta \right. \\ &\quad \left. + A(\theta_0, \varepsilon) \right) + CT \left(B(u, \varepsilon, \theta_0) + C_{\theta_0} \sigma^2(\varepsilon) K'_\theta \right. \\ &\quad \left. + C(u, \theta_0, \varepsilon) \right), \end{aligned}$$

where $C = \max(K, K')$, $K_\theta = \int_{\varepsilon \leq |z| \leq 1} |z|^2 e^{\theta z} \ell(dz)$, $A(\theta_0, \varepsilon) = \left| \int_{|z| < \varepsilon} z(e^{\theta_0 z} - 1) \ell(dz) \right|$, $B(u, \varepsilon, \theta_0) = \left| \int_{|z| \leq \varepsilon} (\sin(uz) - uz) e^{\theta_0 z} \ell(dz) \right|$, $K'_\theta = \int_{|z| \geq \varepsilon} |-\sin(uz) + uz \mathbf{1}_{|z| < 1}| |z| e^{\theta z} \ell(dz) + \int_{|z| \geq \varepsilon} |\cos(uz) - 1| |z| e^{\theta z} \ell(dz)$, and $C(u, \theta_0, \varepsilon) = \left| \int_{|z| \leq \varepsilon} (-\cos(uz) + 1) e^{\theta_0 z} \ell(dz) \right|$. Moreover $A(\theta_0, \varepsilon) \leq \sigma^2(\varepsilon) e^{|\theta|} |\theta_0|$, $B(u, \theta_0, \varepsilon) \leq e^{|\theta_0|} \varepsilon \frac{|u|^3}{2} \sigma^2(\varepsilon)$, and $C(u, \theta_0, \varepsilon) \leq \frac{u^2}{2} e^{|\theta_0|} \sigma^2(\varepsilon)$. Therefore the result follows. \square

The next proposition tells us that also the delta of the option price converges.

Proposition 3.2.2. *Under the condition $u\hat{f}(u) \in L^1(\mathbb{R})$, we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial x} \tilde{\mathbb{E}}_{\theta_\varepsilon} [f(x + L_\varepsilon(T))] = \frac{\partial}{\partial x} \tilde{\mathbb{E}}_{\theta_0} [f(x + L(T))].$$

Proof. We differentiate the integrand in (3.18) and dominate it uniformly in x ,

$$\begin{aligned} \left| \frac{\partial}{\partial x} e^{-iux} \phi_{L_\varepsilon(T)}(-u) \hat{f}(u) \right| &= | -iue^{-iux} \phi_{L_\varepsilon(T)}(-u) \hat{f}(u) | \\ &\leq |u\hat{f}(u)|. \end{aligned}$$

Then, by Prop. 2.27 in Folland [33], we can take the derivative operator inside the integral to get

$$\frac{\partial}{\partial x} \tilde{\mathbb{E}}_{\theta_\varepsilon} [f(x + L_\varepsilon(T))] = \frac{1}{2\pi} \int_{\mathbb{R}} -iue^{-iux} \phi_{L_\varepsilon(T)}(-u) \hat{f}(u) du.$$

Dominating the integrand in the last expression uniformly in ε , the result follows by Prop. 2.24 in Folland [33]. \square

Note that we may derive a similar rate of convergence for the delta as we find for the option prices in Prop. 3.2.1.

The integrability restriction in the proposition above excludes many interesting examples of functions f , like for instance the payoff from a call option. However, we can easily deal with this situation by introducing a damped function f in the following manner. Define for $\alpha > 0$ the function

$$g_\alpha(y) = e^{-\alpha y} f(y). \quad (3.19)$$

Assuming that $g_\alpha \in L^1(\mathbb{R})$ and $\widehat{g}_\alpha \in L^1(\mathbb{R})$ for some $\alpha > 0$, we can apply the above results for g_α . To translate to f , observe that

$$f(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{(\alpha - iu)y} \widehat{g}_\alpha(u) du,$$

and

$$\widehat{g}_\alpha(u) = \widehat{f}(u + i\alpha).$$

Hence, Prop. 3.2.2 holds for any f such that there exists $\alpha > 0$ for which we have the following assumptions

$$(\alpha - iu)\widehat{f}(u + i\alpha) \in L^1(\mathbb{R}) \quad (3.20)$$

and

$$e^{\alpha y} \widetilde{p}_{L(T)} \in L^1(\mathbb{R}).$$

In Chapter 4 we give some examples of payoff functions f satisfying the condition (3.20). In the following we consider an example to illustrate our findings on approximations.

Example. Let us assume that L is an NIG-Lévy process, that is, a Lévy process with NIG-distributed increments. Suppose $L(1)$ is NIG distributed with parameters $\mu \in \mathbb{R}$, $\delta > 0$, $\alpha > 0$, $-\alpha \leq \beta \leq \alpha$. We denote by $L(1) \sim \text{NIG}(\mu, \delta, \alpha, \beta)$. The density is (see Barndorff-Nielsen [14])

$$p_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x - \mu)^2}\right)}{\sqrt{\delta^2 + (x - \mu)^2}}. \quad (3.21)$$

Here, K_1 is the modified Bessel function of the second order with parameter 1, which can be represented by the integral

$$K_1(z) = \frac{\sqrt{\pi}z}{2\Gamma(\frac{3}{2})} \int_1^\infty e^{-zt}(t^2 - 1)^{\frac{1}{2}} dt,$$

for $z > 0$. The cumulant function is

$$G(\theta) = \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta^2 + \theta^2)}\right) + \mu\theta \quad (3.22)$$

which exists for

$$-\alpha - \beta \leq \theta \leq \alpha - \beta.$$

The Lévy measure ℓ is given by

$$\ell(z) = \frac{\delta\alpha}{\pi} e^{\beta z} |z|^{-1} K_1(\alpha|z|). \quad (3.23)$$

In this case $L(t) \sim \text{NIG}(\mu t, \delta t, \alpha, \beta)$ for all $t > 0$. If $0 < \alpha < \frac{1}{2}$ or $\alpha \geq \frac{1}{2}$, $|\mu - r| > \delta\sqrt{2\alpha - 1}$, then the Esscher parameter does not exist, however, Hubalek and Sgarra [45] compute analytically the Esscher parameter in the case $\alpha \geq \frac{1}{2}$, $|\mu - r| \leq \delta\sqrt{2\alpha - 1}$,

$$\theta_0 = -\beta - \frac{1}{2} - \frac{(\mu - r)}{2\delta} \sqrt{\frac{4\alpha^2\delta^2}{(\mu - r)^2 + \delta^2} - 1}. \quad (3.24)$$

Considering the Lévy process L_ε , the Esscher parameter θ_ε exists for $-\alpha - \beta \leq \theta_\varepsilon \leq \alpha - \beta - 1$. To compute the parameter θ_ε , we consider the fact that

$$G(\theta) = G_\varepsilon(\theta) + \int_{|z| < \varepsilon} (e^{\theta z} - 1 - z\theta)\ell(dz) - \frac{1}{2}\theta^2\sigma^2(\varepsilon),$$

which leads to

$$G_\varepsilon(\theta_\varepsilon + 1) - G_\varepsilon(\theta_\varepsilon) = G(\theta_\varepsilon + 1) - G(\theta_\varepsilon) + \sigma^2(\varepsilon)(\theta_\varepsilon + \frac{1}{2}) + \int_{|z| < \varepsilon} (e^{\theta_\varepsilon z}(1 - e^z) + z)\ell(dz).$$

The equation (3.9) is therefore equivalent to

$$G(\theta_\varepsilon + 1) - G(\theta_\varepsilon) = r - \sigma^2(\varepsilon)(\theta_\varepsilon + \frac{1}{2}) - \int_{|z| < \varepsilon} (e^{\theta_\varepsilon z}(1 - e^z) + z)\ell(dz).$$

As $\int_{|z| < \varepsilon} (e^{\theta_\varepsilon z}(1 - e^z) + z)\ell(dz) \simeq -\theta_\varepsilon\sigma^2(\varepsilon)$, we find that θ_ε is approximately the solution of the following equation

$$G(\theta_\varepsilon + 1) - G(\theta_\varepsilon) = r - \frac{1}{2}\sigma^2(\varepsilon).$$

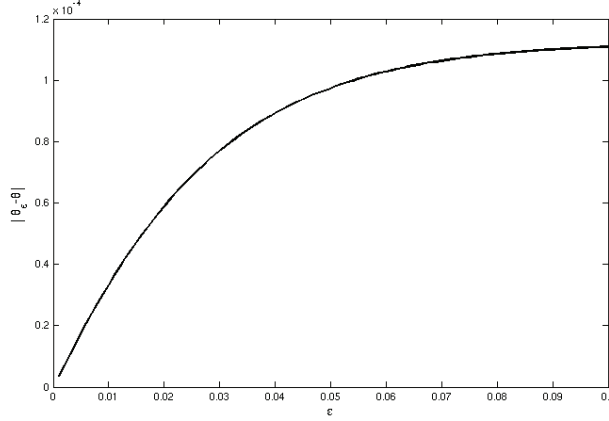
Therefore, if $0 < \alpha < \frac{1}{2}$ or $\alpha \geq \frac{1}{2}$, $|\mu - r + \frac{1}{2}\sigma^2(\varepsilon)| > \delta\sqrt{2\alpha - 1}$, then the Esscher parameter does not exist. If $\alpha \geq \frac{1}{2}$, $|\mu - r + \frac{1}{2}\sigma^2(\varepsilon)| \leq \delta\sqrt{2\alpha - 1}$ then using the expression of $G(\theta)$ in (3.22), we get

$$\theta_\varepsilon = -\beta - \frac{1}{2} - \frac{(\mu + \frac{1}{2}\sigma^2(\varepsilon) - r)}{2\delta} \sqrt{\frac{4\alpha^2\delta^2}{(\mu + \frac{1}{2}\sigma^2(\varepsilon) - r)^2 + \delta^2} - 1}.$$

Moreover, we have that the error becomes

$$\begin{aligned} |\theta_\varepsilon - \theta_0| &= \left| \frac{\mu - r}{2\delta} \left(\sqrt{\frac{4\alpha^2\delta^2}{(\mu + \frac{1}{2}\sigma^2(\varepsilon) - r)^2 + \delta^2} - 1} - \sqrt{\frac{4\alpha^2\delta^2}{(\mu - r)^2 + \delta^2} - 1} \right) \right. \\ &\quad \left. + \frac{\sigma^2(\varepsilon)}{\delta} \sqrt{\frac{4\alpha^2\delta^2}{(\mu + \frac{1}{2}\sigma^2(\varepsilon) - r)^2 + \delta^2} - 1} \right|. \end{aligned}$$

For a concrete numerical example, let $\alpha = 80$, $\beta = \mu = r = 0$, and $\delta = 0.03$. The choice of α and δ here are on the scale relevant for stock prices observed in markets (see for example the estimates in Benth [6] for the NASDAQ and FTSE indices). Figure 3.1 plots the error $|\theta_\varepsilon - \theta_0|$ as a function of ε for $0 < \varepsilon < 0.1$. As we can see, it decays fastly to

Figure 3.1: The variation of the error as a function of ε

zero, in accordance with our expectations. Even for relatively large ε , the error is rather small. This may be attributed to the fact that an NIG distribution with $\mu = \beta = 0$ is symmetric, and very similar to a normal distribution near its center. Notice that in our case the error is analytically given as

$$|\theta_\varepsilon - \theta_0| = \left| \frac{\sigma^2(\varepsilon)}{\delta} \sqrt{\frac{4\alpha^2\delta^2}{\frac{1}{4}\sigma^4(\varepsilon) + \delta^2} - 1} \right|.$$

Therefore, since $0 \leq \sigma^2(\varepsilon) \leq \sigma^2(1)$, we have

$$\frac{|\theta_\varepsilon - \theta_0|}{\sigma^2(\varepsilon)} \in \left(\frac{1}{\delta} \sqrt{\frac{4\alpha^2\delta^2}{\frac{1}{4}\sigma^4(1) + \delta^2} - 1}, \frac{1}{\delta} \sqrt{4\alpha^2 - 1} \right).$$

For our choice of parameters, the interval is very narrow and given by

$$\frac{|\theta_\varepsilon - \theta_0|}{\sigma^2(\varepsilon)} \in (0.8333, 0.8334),$$

for $0 \leq \varepsilon \leq 0.1$. Thus, for practical purposes we have an exact error rate rather than an upper bound.

3.2.2 The minimal entropy martingale measure

The relative entropy $I_{\mathbb{P}}(\widehat{\mathbb{P}})$ of the measure $\widehat{\mathbb{P}}$ with respect to \mathbb{P} is defined by

$$I_{\mathbb{P}}(\widehat{\mathbb{P}}) = \begin{cases} \mathbb{E}_{\widehat{\mathbb{P}}}[\log \frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}] = \mathbb{E}_{\mathbb{P}}[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} \log \frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}], & \text{if } \widehat{\mathbb{P}} \ll \mathbb{P} \\ \infty, & \text{otherwise.} \end{cases}$$

The minimal entropy martingale measure is the probability measure that minimizes the value of the function $I_{\mathbb{P}}(\widehat{\mathbb{P}})$ over all $\widehat{\mathbb{P}} \in EMM(\mathcal{P})$. Fujiwara and Miyahara [35] show the existence of the minimal entropy martingale measure for the geometric Lévy process. Moreover, they show that it can be defined by means of the Esscher transform.

Before, we state the theorem by Fujiwara and Miyahara [35], we introduce the following condition on the Lévy process L .

(C): There exists a constant $\theta^* \in \mathbb{R}$ that satisfies the following conditions

- $\int_{|z|>1} e^z e^{\theta^*(e^z-1)} \ell(dz) < \infty,$
-

$$a + \left(\frac{1}{2} + \theta^*\right)b^2 + \int_{|z|\leq 1} \{(e^z - 1)e^{\theta^*(e^z-1)} - z\} \ell(dz) + \int_{|z|>1} (e^z - 1)e^{\theta^*(e^z-1)} \ell(dz) = r. \quad (3.25)$$

The next result is due to Fujiwara and Miyahara [35].

Theorem 3.2.1. *Suppose that the condition C holds.*

1. *We can define a probability measure $\widetilde{\mathbb{P}}$ on \mathcal{F}_T by means of the Esscher transform,*

$$\left. \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \frac{e^{\theta^* \widehat{L}(t)}}{\mathbb{E}[e^{\theta^* \widehat{L}(t)}]} = e^{\theta^* \widehat{L}(t) - b^* t},$$

where \widehat{L} is the process defined by equation (3.1) and

$$b^* = \frac{\theta^*}{2}(1 + \theta^*)b^2 + \theta^*a + \int_{\mathbb{R}} \{e^{\theta^*(e^z-1)} - 1 - \theta^*z\mathbf{1}_{|z|\leq 1}\} \ell(dz).$$

Thus

$$\begin{aligned} \left. \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t} &= \exp \left\{ \theta^* b W(t) - \frac{1}{2}(\theta^*)^2 b^2 t + \theta^* \int_0^t \int_{\mathbb{R}} (e^z - 1) \widetilde{N}(ds, dz) \right. \\ &\quad \left. - t \int_{\mathbb{R}} (e^{\theta^*(e^z-1)} - 1 - \theta^*(e^z - 1)) \ell(dz) \right\}. \end{aligned}$$

2. *The stochastic process L is still a Lévy process under the probability measure $\widetilde{\mathbb{P}}$ and the characteristic triplet is given by, $(b^2, \widetilde{\ell}, \widetilde{a})$, where*

$$\widetilde{\ell}(dz) = e^{\theta^*(e^z-1)} \ell(dz)$$

and

$$\widetilde{a} = \theta^* b^2 + a + \int_{|z|\leq 1} z(e^{\theta^*(e^z-1)} - 1) \ell(dz).$$

Furthermore, the probability measure $\widetilde{\mathbb{P}}$ is in $EMM(\mathcal{P})$.

3. The probability measure $\tilde{\mathbb{P}}$ attains the minimal entropy in $ACLMM(\mathcal{P})$,

$$\min_{\hat{\mathbb{P}} \in ACLMM(\mathcal{P})} I_{\hat{\mathbb{P}}}(\hat{\mathbb{P}}) = I_{\tilde{\mathbb{P}}}(\tilde{\mathbb{P}}).$$

Note that the first expression in the condition \mathcal{C} is the condition for the moment generating function of the process \hat{L} to exist. The second expression is equivalent to the condition (3.6) (martingale condition).

We consider now the Lévy process L_ε which satisfies the following assumption.

(C'): There exists $\theta_\varepsilon^* \in \mathbb{R}$ that satisfies the following conditions

- $\int_{|z|>1} e^z e^{\theta_\varepsilon^*(e^z-1)} \ell(dz) < \infty$,
-

$$\begin{aligned} a + \left(\frac{1}{2} + \theta_\varepsilon^*\right)(b^2 + \sigma^2(\varepsilon)) + \int_{\varepsilon \leq |z| \leq 1} \{(e^z - 1)e^{\theta_\varepsilon^*(e^z-1)} - z\} \ell(dz) \\ + \int_{|z|>1} (e^z - 1)e^{\theta_\varepsilon^*(e^z-1)} \ell(dz) = r. \end{aligned} \quad (3.26)$$

We define a probability measure $\tilde{\mathbb{P}}_{\theta_\varepsilon^*}^\varepsilon$ by means of the Esscher transform as follows.

$$\frac{d\tilde{\mathbb{P}}_{\theta_\varepsilon^*}^\varepsilon}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{e^{\theta_\varepsilon^* \hat{L}_\varepsilon(t)}}{\mathbb{E}[e^{\theta_\varepsilon^* \hat{L}_\varepsilon(t)}]} = e^{\theta_\varepsilon^* \hat{L}_\varepsilon(t) - b_\varepsilon^* t},$$

where

$$\hat{L}_\varepsilon(t) = L_\varepsilon(t) + \frac{1}{2}(b^2 + \sigma^2(\varepsilon))t + \int_0^t \int_{|z| \geq \varepsilon} (e^z - 1 - z) N(ds, dz) \quad (3.27)$$

and

$$b_\varepsilon^* = \frac{\theta_\varepsilon^*}{2}(1 + \theta_\varepsilon^*)(b^2 + \sigma^2(\varepsilon)) + \theta_\varepsilon^* a + \int_{|z| \geq \varepsilon} \{e^{\theta_\varepsilon^*(e^z-1)} - 1 - \theta_\varepsilon^* z \mathbf{1}_{|z| \leq 1}\} \ell(dz).$$

Thus

$$\begin{aligned} \frac{d\tilde{\mathbb{P}}_{\theta_\varepsilon^*}^\varepsilon}{d\mathbb{P}} \Big|_{\mathcal{F}_t} &= \exp \left\{ \theta_\varepsilon^* (bW(t) + \sigma(\varepsilon)B(t)) - \frac{1}{2}(\theta_\varepsilon^*)^2(b^2 + \sigma^2(\varepsilon))t + \theta_\varepsilon^* \int_0^t \int_{|z| > \varepsilon} (e^z - 1) \tilde{N}(ds, dz) \right. \\ &\quad \left. - t \int_{|z| > \varepsilon} (e^{\theta_\varepsilon^*(e^z-1)} - 1 - \theta_\varepsilon^*(e^z - 1)) \ell(dz) \right\}. \end{aligned}$$

By Theorem 3.2.1, the probability measure $\tilde{\mathbb{P}}_{\theta_\varepsilon^*}^\varepsilon$ is the minimal entropy martingale measure for the discounted price process \hat{S}_ε . Moreover, the process L_ε is still a Lévy process under the measure $\tilde{\mathbb{P}}_{\theta_\varepsilon^*}^\varepsilon$ and the characteristic triplet is given by $\left(b^2 + \sigma^2(\varepsilon), \tilde{\ell}_\varepsilon, \tilde{a}_\varepsilon\right)$, where

$$\tilde{\ell}_\varepsilon(dz) = e^{\theta_\varepsilon^*(e^z-1)} \ell(dz),$$

and

$$\tilde{a}_\varepsilon = \theta_\varepsilon^*(b^2 + \sigma^2(\varepsilon)) + a + \int_{\varepsilon \leq |z| \leq 1} z(e^{\theta_\varepsilon^*(e^z-1)} - 1)\ell(dz).$$

We denote by $\tilde{E}_{\theta_\varepsilon^*}[\cdot]$ the expectation with respect to $\tilde{\mathbb{P}}_{\theta_\varepsilon^*}^\varepsilon$.

The existence and uniqueness of the solution of (3.25) is proved by Fujiwara and Miyahara [35]. With the same way, we can prove the existence and uniqueness of θ_ε^* solution of (3.26) for ε fixed in $(0, 1)$ and thus we have the following proposition.

Proposition 3.2.3. *Define*

$$F(\theta^*) = \theta^*b^2 + \int_{|z| \leq 1} (e^z - 1)(e^{\theta^*(e^z-1)} - 1)\ell(dz) + \int_{|z| > 1} (e^z - 1)e^{\theta^*(e^z-1)}\ell(dz),$$

for $\theta^* \in (-\infty, \theta_0^*)$, where

$$\theta_0^* := \sup\{\theta^* \in \mathbb{R}; \int_{|z| > 1} e^z e^{\theta^*(e^z-1)}\ell(dz) < \infty\}.$$

Then there exists a unique constant $\theta^* \in \mathbb{R}$ satisfying (3.25) if and only if

$$r - b_1 \in \begin{cases} \left(\lim_{\theta^* \downarrow -\infty} F(\theta^*), \lim_{\theta^* \uparrow \theta_0^*} F(\theta^*) \right] & \text{in the case when } \theta_0^* < +\infty \\ \left(\lim_{\theta^* \downarrow -\infty} F(\theta^*), \lim_{\theta^* \uparrow \theta_0^*} F(\theta^*) \right) & \text{in the case when } \theta_0^* = +\infty, \end{cases} \quad (3.28)$$

where $b_1 = \frac{1}{2}b^2 + a + \int_{|z| \leq 1} (e^z - 1 - z)\ell(dz)$.

Define now

$$F_\varepsilon(\theta^*) = \theta^*(b^2 + \sigma^2(\varepsilon)) + \int_{\varepsilon \leq |z| \leq 1} (e^z - 1)(e^{\theta^*(e^z-1)} - 1)\ell(dz) + \int_{|z| > 1} (e^z - 1)e^{\theta^*(e^z-1)}\ell(dz),$$

for $\theta^* \in (-\infty, \theta_0^*)$. Then for each $\varepsilon \in (0, 1)$, there exists a unique constant $\theta_\varepsilon^* \in \mathbb{R}$ satisfying (3.26) if and only if

$$r - b_1^\varepsilon \in \begin{cases} \left(\lim_{\theta^* \downarrow -\infty} F_\varepsilon(\theta^*), \lim_{\theta^* \uparrow \theta_0^*} F_\varepsilon(\theta^*) \right] & \text{in the case when } \theta_0^* < +\infty \\ \left(\lim_{\theta^* \downarrow -\infty} F_\varepsilon(\theta^*), \lim_{\theta^* \uparrow \theta_0^*} F_\varepsilon(\theta^*) \right) & \text{in the case when } \theta_0^* = +\infty, \end{cases} \quad (3.29)$$

where $b_1^\varepsilon = \frac{1}{2}(b^2 + \sigma^2(\varepsilon)) + a + \int_{\varepsilon \leq |z| \leq 1} (e^z - 1 - z)\ell(dz)$.

By the same argument as in Lemma 3.2.1 and under the conditions (3.28) and (3.29), we can prove that θ_ε^* is bounded uniformly in ε and that

$$|\theta_\varepsilon^* - \theta^*| \leq C_{\theta^*}\sigma^2(\varepsilon) \quad \text{and} \quad |\theta_\varepsilon^*| \leq |\theta^*| + C_{\theta^*}\sigma^2(1).$$

Thus, we have the following result.

Proposition 3.2.4. *For $\hat{f} \in L^1(\mathbb{R})$, we have*

$$\lim_{\varepsilon \rightarrow 0} \tilde{\mathbb{E}}_{\theta_\varepsilon^*}[f(x + L_\varepsilon(T))] = \tilde{\mathbb{E}}_{\theta^*}[f(x + L(T))].$$

Proof. Recall from (3.18) that

$$\tilde{\mathbb{E}}_{\theta_\varepsilon^*}[f(x + L_\varepsilon(T))] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \phi_{L_\varepsilon(T)}(-u) \hat{f}(u) du. \quad (3.30)$$

From the characteristic triplet of the process L_ε under the measure $\tilde{\mathbb{P}}^\varepsilon$, we can write the characteristic function $\phi_{L_\varepsilon(T)}(u)$ as follows:

$$\phi_{L_\varepsilon(T)}(u) = \exp \left\{ i\tilde{a}_\varepsilon T u - \frac{T}{2}(b^2 + \sigma^2(\varepsilon))u^2 + T \int_{|z| \geq \varepsilon} \{e^{iuz} - 1 - iuz \mathbf{1}_{|z| < 1}\} \tilde{\ell}_\varepsilon(dz) \right\}.$$

As θ_ε^* is bounded and converges to θ^* then $\phi_{L_\varepsilon(T)}(u)$ converges to $\phi_{L(T)}(u)$, for all $u \in \mathbb{R}$, where $\phi_{L(T)}(u)$ is the characteristic function of L under the measure $\tilde{\mathbb{P}}$. Taking the limit inside the integral in equation (3.30), the result follows. \square

Using the same arguments as in Prop 3.2.2, we can also show that the delta of the option price converges and we have the following proposition.

Proposition 3.2.5. *Under the condition $u\hat{f}(u) \in L^1(\mathbb{R})$, we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial x} \tilde{\mathbb{E}}_{\theta_\varepsilon^*}[f(x + L_\varepsilon(T))] = \frac{\partial}{\partial x} \tilde{\mathbb{E}}_{\theta^*}[f(x + L(T))].$$

Note that we may derive convergence rates as well for these two results, analogous to the Esscher transform case.

3.2.3 The minimal martingale measure

In this section, we assume that the Lévy measure of the process L satisfies the following integrability conditions

$$\int_{z > 1} e^{2z} \ell(dz) < \infty. \quad (3.31)$$

Recall the dynamics of \hat{S} in (3.2). Since it is a semimartingale, we can decompose it into a local martingale M and a finite variation process A , with $A(0) = 0$, where M and A have the following expressions

$$M(t) = \hat{S}(0) + \int_0^t b \hat{S}(s-) dW(s) + \int_0^t \int_{\mathbb{R}_0} \hat{S}(s-) (e^z - 1) \tilde{N}(ds, dz), \quad (3.32)$$

$$A(t) = \int_0^t (a_1 - r) \hat{S}(s) ds. \quad (3.33)$$

We denote by $\langle X \rangle$ the predictable compensator of the process X , i.e. $X^2(t) - \langle X \rangle(t)$, $0 \leq t \leq T$, is a local martingale. Then, we have

$$\langle M \rangle(t) = \int_0^t b^2 \hat{S}^2(s) ds + \int_0^t \int_{\mathbb{R}_0} \hat{S}^2(s) (e^z - 1)^2 \ell(dz) ds$$

and we can represent the process A as follows

$$A(t) = \int_0^t \frac{a_1 - r}{b^2 \widehat{S}(s) + \widehat{S}(s) \int_{\mathbb{R}_0} (e^z - 1)^2 \ell(dz)} d\langle M \rangle(s). \quad (3.34)$$

Let α be the integrand in equation (3.34), that is, the predictable process given by

$$\alpha(t) := \frac{a_1 - r}{b^2 \widehat{S}(t) + \widehat{S}(t) \int_{\mathbb{R}_0} (e^z - 1)^2 \ell(dz)}, \quad 0 \leq t \leq T. \quad (3.35)$$

We define a process K by means of α as follows

$$K(t) = \int_0^t \alpha^2(s) d\langle M \rangle(s) = \frac{(a_1 - r)^2}{b^2 + \int_{\mathbb{R}_0} (e^z - 1)^2 \ell(dz)} t. \quad (3.36)$$

The process K is called the mean-variance trade-off process. The processes defined above will be used later in connection with our analysis of the minimal martingale measure. Under the condition (3.31), the local martingale part M defined in (3.32) is a square integrable \mathbb{P} -martingale and the stock price process \widehat{S} is a special semimartingale (see Schweizer [64]). Moreover, for any $\widetilde{\mathbb{P}} \in ACLMM(\mathbb{P})$, the process \widehat{S} is not only a local martingale but a martingale under $\widetilde{\mathbb{P}}$.

The notion of minimal martingale measure was introduced in Föllmer and Schweizer [37]. A martingale measure $\widetilde{\mathbb{P}}$ is called minimal if any square-integrable \mathbb{P} -martingale which is orthogonal to the martingale part of \widehat{S} under \mathbb{P} remains a martingale under $\widetilde{\mathbb{P}}$. Föllmer and Schweizer [37] show the existence and uniqueness of this measure in the case of special semimartingales. The condition (3.31) ensures the existence and uniqueness of the minimal martingale measure in our model, and we have the next result due to Prop 4.1 in Arai [1].

Theorem 3.2.2. *The following holds;*

1. *We define a probability measure $\widetilde{\mathbb{P}}$ on \mathcal{F}_T by*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left\{ - \int_0^t \alpha(s) dM(s) - \frac{1}{2} K(t) \right\},$$

where the processes α, M and K are defined by equations (3.35), (3.32) and (3.36) respectively. Denote by $\gamma = \alpha(s) \widehat{S}(s) = \frac{a_1 - r}{b^2 + \int_{\mathbb{R}} (e^z - 1)^2 \ell(dz)}$. Then

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left\{ - \int_0^t \gamma b dW(s) - \int_0^t \int_{\mathbb{R}_0} \gamma (e^z - 1) \widetilde{N}(ds, dz) - \frac{1}{2} K(t) \right\}.$$

2. *The stochastic process L is still a Lévy process under the probability measure $\widetilde{\mathbb{P}}$ and the characteristic triplet is given by $(b^2, \widetilde{\ell}, \widetilde{a})$, where*

$$\widetilde{\ell}(dz) = \{(e^z - 1)\gamma + 1\} \ell(dz)$$

and

$$\widetilde{a} = a + b^2 \gamma + \int_{|z| \leq 1} \gamma z (e^z - 1) \ell(dz).$$

3. The density process $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ is a square integrable \mathbb{P} -martingale.

4. The measure $\tilde{\mathbb{P}}$ is a minimal martingale measure.

We define γ_ε by

$$\gamma_\varepsilon = \frac{c_\varepsilon}{b^2 + \sigma^2(\varepsilon) + \int_{|z| \geq \varepsilon} (e^z - 1)^2 \ell(dz)}, \quad (3.37)$$

where

$$c_\varepsilon = a + \frac{1}{2}(b^2 + \sigma^2(\varepsilon)) + \int_{|z| \geq \varepsilon} (e^z - 1 - z\mathbf{1}_{|z| \leq 1}) \ell(dz) - r. \quad (3.38)$$

Let $\tilde{\mathbb{P}}^\varepsilon$ be a measure which is absolutely continuous with respect to \mathbb{P} such that

$$\left. \frac{d\tilde{\mathbb{P}}^\varepsilon}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp \left\{ - \int_0^t \gamma_\varepsilon \sqrt{b^2 + \sigma^2(\varepsilon)} d\tilde{B}_\varepsilon(s) - \int_0^t \int_{|z| > \varepsilon} \gamma_\varepsilon (e^z - 1) \tilde{N}(ds, dz) - \frac{1}{2} K_\varepsilon(t) \right\}, \quad (3.39)$$

where

$$\tilde{B}_\varepsilon(t) = \frac{b}{\sqrt{b^2 + \sigma^2(\varepsilon)}} W(t) + \frac{\sigma(\varepsilon)}{\sqrt{b^2 + \sigma^2(\varepsilon)}} B(t)$$

and

$$K_\varepsilon(t) = \frac{c_\varepsilon^2}{b^2 + \sigma^2(\varepsilon) + \int_{|z| > \varepsilon} (e^z - 1)^2 \ell(dz)}.$$

By Theorem 3.2.2 and under the condition (3.31), the measure $\tilde{\mathbb{P}}^\varepsilon$ is a minimal martingale measure for the discounted price process \tilde{S}_ε . Moreover, the stochastic process L_ε is still a Lévy process under $\tilde{\mathbb{P}}^\varepsilon$ and the characteristic triplet is given by $(b^2 + \sigma^2(\varepsilon), \tilde{\ell}_\varepsilon, \tilde{a}_\varepsilon)$, where

$$\tilde{\ell}_\varepsilon(dz) = \{(e^z - 1)\gamma_\varepsilon + 1\} \mathbf{1}_{|z| \geq \varepsilon} \ell(dz), \quad (3.40)$$

and

$$\tilde{a}_\varepsilon = a + (b^2 + \sigma^2(\varepsilon))\gamma_\varepsilon + \int_{\varepsilon \leq |z| \leq 1} \gamma_\varepsilon z (e^z - 1) \ell(dz). \quad (3.41)$$

Denote by $\tilde{\mathbb{E}}_\varepsilon[\cdot]$ the expectation with respect to the measure $\tilde{\mathbb{P}}^\varepsilon$.

We have the following convergence result.

Proposition 3.2.6. *For $\hat{f} \in L^1(\mathbb{R})$, we have*

$$\lim_{\varepsilon \rightarrow 0} \tilde{\mathbb{E}}_\varepsilon[f(x + L_\varepsilon(T))] = \tilde{\mathbb{E}}[f(x + L(T))].$$

Proof. Recall from (3.18) that

$$\tilde{\mathbb{E}}_\varepsilon[f(x + L_\varepsilon(T))] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \phi_{L_\varepsilon(T)}(u) \hat{f}(u) du. \quad (3.42)$$

From the characteristics of the process L_ε under the measure $\tilde{\mathbb{P}}^\varepsilon$, we can prove that the characteristic function $\phi_{L_\varepsilon(T)}(u)$ converges to $\phi_{L(T)}(u)$, for all $u \in \mathbb{R}$, where $\phi_{L(T)}(u)$ is the characteristic function of L under the measure $\tilde{\mathbb{P}}$. Taking the limit inside the integral in equation (3.42), we get the result. \square

Remark. Let us define $D(\hat{\mathbb{P}}, \mathbb{P})$ as

$$D(\hat{\mathbb{P}}, \mathbb{P}) = \sqrt{\text{Var}\left(\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right)}. \quad (3.43)$$

A probability measure $\tilde{\mathbb{P}}$ is the *mean-variance martingale measure* if it minimizes $D(\hat{\mathbb{P}}, \mathbb{P})$ over all $\hat{\mathbb{P}} \in \text{ACLLM}(\mathcal{P})$. In Theorem 8 in Schweizer [64], it is shown that the mean-variance martingale measure coincides with the minimal martingale measure if the following conditions hold:

- The price process \hat{S} is decomposed into a martingale process and a finite variation process.
- The finite variation process A is absolutely continuous with respect to $\langle M \rangle$.
- The mean-variance trade-off process K is deterministic.

In our model, these conditions are satisfied. Our convergence results for the minimal martingale measure transfer to the mean-variance measure as well.

Lévy models robustness and sensitivity

This chapter is extracted from the paper "Lévy models robustness and sensitivity" by Fred Espen Benth, Giulia Di Nunno, and Asma Khedher, published in QP-PQ: Quantum Probability and White Noise Analysis, Proceedings of the 29th Conference in Hammamet, Tunisia, 1318 October 2008. H. Ouerdiane and A Barhoumi (eds.), World Scientific, **25**, (2010) 153–184.

In this chapter, we study the robustness of the sensitivity with respect to parameters in expectation functionals with respect to various approximations of a Lévy process. As sensitivity parameter, we focus on the delta of an European option as the derivative of the option price with respect to the current value of the underlying asset. We prove that the delta is stable with respect to natural approximations of a Lévy process, including approximating the small jumps by a Brownian motion. Our methods are based on the density method, and we propose a new *conditional density method* appropriate for our purposes. Several examples are given, including numerical examples demonstrating our results in practical situations.

The chapter is organized as follows. In Section 4.1 we introduce the conditional density method for the computation of the delta. In Section 4.2 we discuss the problems related to model robustness and we present our results in connection to the analysis of sensitivity. Several examples are provided, including different classes of Lévy process and relevant functions f . In Section 4.3 a numerical study investigates our findings in a practical setting based on Monte Carlo simulations. Comments on our results are given as conclusions in Section 4.4.

4.1 Conditional density method for the computation of derivatives

In this section, we propose a conditional density approach which may be useful in certain contexts. We consider the situation where we have two independent strictly positive random variables Y and Z with densities p_Y and p_Z , respectively. In some situations that we will encounter in the sequel, only one of the two densities may be known, or one of

the two may be simpler to be used for computational purposes. We propose a *conditional density method* for such cases.

Obviously, if the density of $Y + Z$ is known, we are in the situation described in Section 2.1, Chapter 2. Under the hypotheses stated there, we may apply the standard density method in order to find the derivative of the functional

$$F(x) = \mathbb{E}[f(x + Y + Z)] .$$

In this case we find

$$F'(x) = F'_{Y+Z}(x) = \mathbb{E}[f(x + Y + Z)(-\partial \ln p_{Y+Z}(Y + Z))] .$$

See equation (2.9). We use the notation $F'_{Y+Z}(x)$ to emphasize that we apply the density method to the sum $Y + Z$.

On the other hand, if only one of the two densities p_Y or p_Z is known or better fitting the computations, we can apply the *conditional density method* as follows. Since by conditioning we have

$$F(x) = \mathbb{E}[\mathbb{E}[f(x + Y + Z) | Y]] = \mathbb{E}[\mathbb{E}[f(x + Y + Z) | Z]] ,$$

we find (see Sato [59], Prop. 1.16)

$$F(x) = \int_{\mathbb{R}} \mathbb{E}[f(y + Z)] p_Y(y - x) dy = \int_{\mathbb{R}} \mathbb{E}[f(z + Y)] p_Z(z - x) dz .$$

This holds as long as $\mathbb{E}[f(\cdot + Z)] p_Y(\cdot - x)$ is integrable (or, symmetrically, $\mathbb{E}[f(\cdot + Y)] p_Z(\cdot - x)$ is integrable). Strictly speaking, the Proposition 1.16 in Sato [59] is only valid under boundedness conditions, however, these can be relaxed by standard limiting arguments. The expressions of $F(x)$ can be used in two ways to derive the derivative $F'(x)$. First, we find

$$F'_Y(x) = \mathbb{E}[f(x + Y + Z)(-\partial \ln p_Y(Y))] ,$$

as long as p_Y is differentiable and $\mathbb{E}[f(\cdot - Z)] p'_Y(\cdot - x)$ is bounded by an integrable function uniformly in x , say. Symmetrically, we obtain

$$F'_Z(x) = \mathbb{E}[f(x + Y + Z)(-\partial \ln p_Z(Z))] ,$$

whenever $\mathbb{E}[f(\cdot - Y)] p'_Z(\cdot - x)$ is bounded by an integrable function uniformly in x . Obviously, $F'_Y(x) = F'_Z(x) = F'_{Y+Z}(x) = F'(x)$, however, the three different calculations lead to three different weights, being, respectively,

$$\begin{aligned} \pi_{Y+Z} &:= -\partial \ln p_{Y+Z}(Y + Z) \\ \pi_Y &:= -\partial \ln p_Y(Y) \\ \pi_Z &:= -\partial \ln p_Z(Z) . \end{aligned}$$

The last two weights are resulting from the conditional density method, while the first one is from the density method. These three weights are genuinely different.

4.2 Robustness of the delta to model choice

In this section we analyze the sensitivity of expectation functionals with respect to Lévy processes and their approximations. Our main focus will be on cases where a Lévy process L and its approximation L_ε are indistinguishable in practical contexts for small ε . Hence, in a concrete application, we may think of two models L and L_ε for the same random phenomenon which we cannot in practical terms separate. For instance, we may think of two speculators in a financial market who want to price an option. The first investor believes in a model given by L , while the other chooses a model L_ε , being slightly different than the former. The distributions of the two models will be very close, and thus also the derived option price. However, the main question we want to analyze in this chapter is whether the same holds true for the sensitivities (or the Greeks in financial terminology). We refer to this question as a problem of robustness of sensitivities to model choice.

Recall the processes L and L_ε given respectively in equations (2.1) and (2.3). Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is a measurable function and that for each x belonging to a compact set of \mathbb{R} , there exists a random variable $U \in L^1(\mathbb{P})$ such that $|f(x + L_\varepsilon(1))| \leq U$ for all ε . Without loss of generality, we can consider $x \in [x_1, x_2]$, for some $x_1, x_2 \in \mathbb{R}$. Since $f(x + L_\varepsilon(1))$ converges almost surely to $f(x + L(1))$, by dominated convergence it holds that

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[f(x + L_\varepsilon(1))] = \mathbb{E}[f(x + L(1))] = F(x). \quad (4.1)$$

Such expectation functionals arise in pricing of options, where f is the payoff function from the option and $x + L(1)$ is the state of the underlying asset at exercise time 1. If $f(x) = \mathbf{1}_{\{x < q\}}$, we may view the expectation as coming from a simple quantile risk measure on the random variable $x + L(1)$, where the x is the initial state of the system under consideration. For notational simplification, we introduce

$$F_\varepsilon(x) := \mathbb{E}[f(x + L_\varepsilon(1))] \quad (4.2)$$

and set $F_0(x) = F(x)$. We analyze $F'_\varepsilon(x)$ and its convergence to $F'(x)$.

To differentiate $F_\varepsilon(x)$, we have in fact a multiple of different approaches. Motivated from the Malliavin method of Davis and Johansson [24] for jump diffusions, it is natural to use the conditional density method with respect to the Brownian motion. However, this leads to three possibilities. Either we can differentiate with respect to the original Brownian motion W , or with respect to B , or finally with respect to a *new* Brownian motion defined as the sum of the two. This will lead to three new expectation operators, which we would like to converge to $F'(x)$ when $\varepsilon \downarrow 0$. As we will see, this is indeed the case, however, the three choices have different properties. Obviously, if the distribution of L_ε is available, one would prefer to use the density method directly on this. However, due to the truncation of the jumps at ε , this will not, in most practical applications, be available.

To get some intuition on the problem we are facing, let us consider a trivial case of a Brownian motion

$$L(t) = bW(t).$$

Of course, in this case we have by a straightforward application of the density method

$$F'(x) = \mathbb{E} \left[f(x + bW(1)) \frac{W(1)}{b} \right].$$

Now, introduce

$$L_\varepsilon(t) = \sqrt{b^2 - \varepsilon^2}W(t) + \varepsilon B(t)$$

where B is an independent Brownian motion. Note that, in distribution, L_ε is identical to L . So in this sense it is not an approximation of L as for the Lévy case. However, we are mimicking the approximation procedure above by representing the “small jumps” of W by a new Brownian motion B . If we first apply the conditional density method on W , we find

$$\begin{aligned} F'_\varepsilon(x) &= \frac{d}{dx} \int_{\mathbb{R}} \mathbb{E} [f(u + \varepsilon B(1))] p_{\sqrt{b^2 - \varepsilon^2}W(1)}(u - x) dx \\ &= \mathbb{E} \left[f(x + L_\varepsilon(1)) \frac{W(1)}{\sqrt{b^2 - \varepsilon^2}} \right], \end{aligned}$$

where we recall that p_X denotes the probability density of the random variable X . We see easily that $F'_\varepsilon(x)$ converges nicely to $F'(x)$. Next, differentiating using the distribution of $\sqrt{b^2 - \varepsilon^2}W(1) + \varepsilon B(1)$ leads similarly to

$$F'_\varepsilon(x) = \mathbb{E} \left[f(x + L_\varepsilon(1)) \frac{\sqrt{b^2 - \varepsilon^2}W(1) + \varepsilon B(1)}{b^2} \right].$$

This will again converge nicely to $F'(x)$. Finally, apply the procedure with respect to $B(t)$ to find

$$F'_\varepsilon(x) = \mathbb{E} \left[f(x + L_\varepsilon(1)) \frac{B(1)}{\varepsilon} \right].$$

This results in a sensitivity weight $B(1)/\varepsilon$ which explodes when $\varepsilon \downarrow 0$, and it is not immediately clear by direct inspection of the functional if it is nicely behaving when taking the limit. However, since all the three approaches above lead to the same derivative $F'_\varepsilon(x)$, we are ensured that the limit also in this case is equal to $F'(x)$. However, from a practical perspective the weight will have a very high variance compared to the two first approaches, and therefore it is not useful in numerical simulations. This illustrates that the approach of Davis and Johansson [24] is not necessarily leading to sensitivity weights which are “good”. In particular we should notice that in the case of pure-jump Lévy processes we will not have any W -term, and we are somehow “forced” to use the density method with respect to B , the approximating Brownian motion. This may lead to problems when understanding the limit since we will not have any comparison.

Let us go back to the general case, where we first suppose that $b > 0$ in (2.1) and apply the density method on the combination of W and B . To distinguish between the different sensitivity weights, we introduce some notation. Let the derivative of $F_\varepsilon(x)$ with respect to x resulting from applying the density method on W , which is the Brownian motion in the Lévy-Kintchine representation of L , be denoted by $F'_{\varepsilon,W}(x)$. Further, we use the notation $F'_{\varepsilon,B}(x)$ and $F'_{\varepsilon,B,W}(x)$ for the derivative when we use the density method with

respect to the small-jump approximating process B or $bW + \sigma(\varepsilon)B$, respectively. Note that even though we may have the density of L , it may be very hard to find the density of L_ε , and thus to apply the density method on the approximating process directly.

We denote by C_b^k the space of k -times continuously differentiable functions with all derivatives bounded, C_b^0 will be denoted by C_b , the space of bounded and continuous functions. It is simple to derive the following result:

Proposition 4.2.1. *Suppose $f \in C_b$. For every $\varepsilon > 0$, we have that*

$$\begin{aligned} F'_{\varepsilon,B}(x) &= \mathbb{E} \left[f(x + L_\varepsilon(1)) \frac{B(1)}{\sigma(\varepsilon)} \right] \\ F'_{\varepsilon,B,W}(x) &= \mathbb{E} \left[f(x + L_\varepsilon(1)) \frac{bW(1) + \sigma(\varepsilon)B(1)}{b^2 + \sigma^2(\varepsilon)} \right]. \end{aligned}$$

If $b > 0$, we have in addition that

$$F'_{\varepsilon,W}(x) = \mathbb{E} \left[f(x + L_\varepsilon(1)) \frac{W(1)}{b} \right].$$

Proof. Using the conditional density method applied to B , we get

$$F'_\varepsilon(x) = \frac{\partial}{\partial x} \int_{\mathbb{R}} \mathbb{E}[f(u + a + bW(1) + Z(1) + \tilde{Z}_\varepsilon(1))] p_{\sigma(\varepsilon)B(1)}(u - x) dx.$$

Here we can dominate the density $P_{\sigma(\varepsilon)B(1)}(u - x)$ uniformly in x by an integrable function which is a sufficient condition to take the derivative inside the integral if f is bounded. Applying the conditional density method to $bW + \sigma(\varepsilon)B$ and W , respectively, and using the same arguments above to take the derivative inside the integral, we get the result. \square

Note that $F'_{\varepsilon,B}(x) = F'_{\varepsilon,B,W}(x) = F'_{\varepsilon,W}(x)$ for all $\varepsilon > 0$. Moreover, we have the following robustness result when $b > 0$, that is, when the Lévy process L has a continuous martingale term.

Proposition 4.2.2. *Suppose that the diffusion coefficient $b > 0$ and that $f \in C_b$. Then we have*

$$\lim_{\varepsilon \downarrow 0} F'_{\varepsilon,W}(x) = \lim_{\varepsilon \downarrow 0} F'_{\varepsilon,B,W}(x) = \lim_{\varepsilon \downarrow 0} F'_{\varepsilon,B}(x) = \mathbb{E} \left[f(x + L(1)) \frac{W(1)}{b} \right] = F'(x).$$

Proof. This hinges on the fact that,

$$F'(x) = \mathbb{E} \left[f(x + L(1)) \frac{W(1)}{b} \right].$$

Now, by the assumption on $f(x + L_\varepsilon(1))W(1)$ and the dominated convergence theorem, we find that

$$\lim_{\varepsilon \downarrow 0} F'_{\varepsilon,W}(x) = F'(x).$$

Furthermore, since $F'_{\varepsilon,B}(x) = F'_{\varepsilon,B,W}(x) = F'_{\varepsilon,W}(x)$, we have that the limit of $F'_{\varepsilon,B,W}(x)$ and $F'_{\varepsilon,B}(x)$ also exist and are equal to $F'(x)$. This proves the result. \square

Remark that although we cannot bound $B(1)/\sigma(\varepsilon)$ by some integrable random variable, we still obtain the convergence. This depends on the fact that the derivative $F'_{\varepsilon,B}(x)$ is equal to $F'_{\varepsilon,W}(x)$ when $b > 0$. When $b = 0$, we can not use this argument anymore, however, we have the following simple result when f is smooth.

Proposition 4.2.3. *Suppose $f \in C_b^1$ and that there exists a random variable $U \in L^1(\mathbb{P})$ such that $|f'(x + L_\varepsilon(1))| \leq U$ uniformly in x and ε . Then*

$$\lim_{\varepsilon \rightarrow 0} F'_{\varepsilon,B}(x) = F'(x) = \mathbb{E}[f'(x + L(1))] .$$

Proof. First, observe that $|f'(x + L(1))| \leq U$ uniformly in x by choosing $\varepsilon = 0$ in the assumption. Hence, by Thm. 2.27 in Folland [33], $F(x)$ is differentiable, and we can move the differentiation inside the expectation operator to obtain

$$F'(x) = \mathbb{E}[f'(x + L(1))] .$$

This proves the second equality. Next, by the same argument, we have that

$$F'_\varepsilon(x) = \frac{d}{dx} \mathbb{E}[f(x + L_\varepsilon(1))] = \mathbb{E}[f'(x + L_\varepsilon(1))] .$$

From the conditional density method, we know that $F'_{\varepsilon,B}(x) = F'_\varepsilon(x)$. By dominated convergence, it holds that

$$\lim_{\varepsilon \rightarrow 0} F'_\varepsilon(x) = F'(x)$$

and the proof is complete. \square

Note that the result holds for all $b \geq 0$, and we could have used it to prove the limits for $b > 0$ as well in the smooth case of f .

In many applications, like for instance in finance, the assumption that f should be continuous and bounded is too restrictive. For example, a call option will lead to an unbounded function, whereas a digital option gives a discontinuous f . Hence, it is natural to look for extensions of the above results to classes of functions where the conditions on f are weakened. One natural approach is to look at classes of functions f which can be approximated by functions in C_b . Another path, which we shall take here, is to apply Fourier methods.

Let now $f \in L^1(\mathbb{R})$. From equation (3.18), it follows that

$$\mathbb{E}[f(x + L(1))] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \phi_{L(1)}(-u) \widehat{f}(u) du , \quad (4.3)$$

where $\phi_{L(1)}$ is the characteristic function of $L(1)$ defined from the Lévy-Kintchine formula as

$$\phi_{L(1)}(u) = \exp \left(iau - \frac{1}{2} b^2 u^2 + \int_{\mathbb{R}_0} e^{iuz} - 1 - iuz 1_{|z| < 1} \ell(dz) \right) . \quad (4.4)$$

We have the following Lemma for the delta.

Lemma 4.2.1. *Under the condition $u\hat{f}(u) \in L^1(\mathbb{R})$ we have*

$$F'(x) = \frac{\partial}{\partial x} \mathbb{E}[f(x + L(1))] = \frac{1}{2\pi} \int_{\mathbb{R}} -iue^{-iux} \phi_{L(1)}(-u) \hat{f}(u) du.$$

Proof. We differentiate the integrand in (4.3) and dominate it uniformly in x :

$$\left| \frac{\partial}{\partial x} e^{-iux} \phi_{L(1)}(-u) \hat{f}(u) \right| = \left| -iu \frac{e^{-iux}}{2\pi} \phi_{L(1)}(-u) \hat{f}(u) \right| \leq |u \hat{f}(u)|.$$

The result follows by appealing to Prop. 2.27 in Folland [33].

Note that the condition $u\hat{f}(u) \in L^1(\mathbb{R})$ is related to the derivative of f , since as long as f is differentiable we have $\hat{f}'(u) = u\hat{f}(u)$ whenever $f' \in L^1(\mathbb{R})$. We finally reach the desired stability result for non-smooth f 's.

Proposition 4.2.4. *Suppose that*

$$u\hat{f}(u) \in L^1(\mathbb{R}).$$

Then we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial x} \mathbb{E}[f(x + L_\varepsilon(1))] = \frac{\partial}{\partial x} \mathbb{E}[f(x + L(1))].$$

Proof. From Lemma 4.2.1 applied to $L_\varepsilon(1)$ we have

$$\frac{\partial}{\partial x} \mathbb{E}[f(x + L_\varepsilon(1))] = \frac{1}{2\pi} \int_{\mathbb{R}} -iue^{-iux} \phi_{L_\varepsilon(1)}(-u) \hat{f}(u) du.$$

But,

$$|-iue^{-iux} \phi_{L_\varepsilon(1)}(-u) \hat{f}(u)| \leq |u \hat{f}(u)|$$

which, from the assumption, permits us to take the limit inside the integral and the result follows by Prop. 2.24 in Folland [33].

Observe that in the Proposition above we handle $b \geq 0$, and there is no need to differentiate between the cases $b = 0$ and $b > 0$. There is no requirement of continuity of f in the above arguments. By introducing a damped function as in (3.19), we have that Prop. 4.2.4 holds for any f such that there exists $\alpha > 0$ for which we have the following assumptions

$$(\alpha - iu)\hat{f}(u + i\alpha) \in L^1(\mathbb{R}) \quad \text{and} \quad e^{\alpha y} p_{L(1)}(dy) \in L^1(\mathbb{R}).$$

As illustration we consider two examples. First, let f be the payoff from a call option written on an asset with price defined as $S(t) = S(0) \exp(L(t))$ ($S(0) > 0$). Then, with $x = \ln S(0)$, we have

$$f(y) = \max(e^y - K, 0)$$

for $K > 0$ being the strike price. For $\alpha > 1$, we have that $g_\alpha \in L^1(\mathbb{R})$. Moreover,

$$\hat{g}_\alpha(u) = \frac{K e^{(iu - \alpha) \ln K}}{(iu - \alpha)(iu - \alpha + 1)},$$

which is in $L^1(\mathbb{R})$. By a direct calculation, we find that

$$(\alpha - iu)\widehat{f}(u + i\alpha) = \frac{K^{1+iu-\alpha}}{1 + iu - \alpha},$$

which belongs to $L^1(\mathbb{R})$. Hence, Prop. 4.2.4 ensures that the approximation $L_\varepsilon(1)$ gives a delta which converges to the delta resulting from the model $L(1)$.

We consider now a digital option written on an asset with price defined as $S(t) = S(0)\exp(L(t))$ ($S(0) > 0$). Then, with $x = \ln S(0)$, we have

$$f(y) = 1_{\{e^y > B\}}, \quad B \in \mathbb{R}_+.$$

For $\alpha > 0$, we have that $g_\alpha \in L^1(\mathbb{R})$. Moreover,

$$\widehat{g}_\alpha(u) = \frac{-B^{iu-\alpha}}{iu - \alpha},$$

which is in $L^1(\mathbb{R})$. By a direct calculation, we find that

$$(\alpha - iu)\widehat{f}(u + i\alpha) = B^{iu-\alpha},$$

which belongs to $L^1(\mathbb{R})$.

4.2.1 Robustness to smoothing of a Lévy process

In the above analysis we have focused entirely on the approximation of the small jumps in a Lévy process. However, we can apply our analysis also in a slightly different situation. Suppose that we are dealing with a Lévy process for which the density (and its log-derivative) may be hard to compute, or may not even be existent analytically. In this case one may approximate the derivative of $F(x)$ by considering the following Lévy process:

$$\widehat{L}_\varepsilon(t) := L(t) + \widehat{\sigma}(\varepsilon)B(t), \quad (4.5)$$

where B is a Brownian motion independent of L and

$$\lim_{\varepsilon \downarrow 0} \widehat{\sigma}(\varepsilon) = 0.$$

We call \widehat{L}_ε a *smoothing* of L , since we add an independent Brownian motion which has a smooth density, and thus \widehat{L}_ε will possess a smooth density as well.

Using the same proof as in Prop. 2.0.1, we have that \widehat{L}_ε converges in $L^1(\mathbb{P})$ to L . Furthermore, since obviously $\widehat{\sigma}(\varepsilon)B(1)$ converges *a.s.* to zero, $\widehat{L}_\varepsilon(1)$ converges *a.s.* to $L(1)$. We have

$$\widehat{F}'_\varepsilon(x) = \mathbb{E} \left[f(x + \widehat{L}_\varepsilon(1)) \frac{B(1)}{\widehat{\sigma}(\varepsilon)} \right]. \quad (4.6)$$

Tracing through the arguments in the preceeding subsection and assuming the right conditions on f , we find that

$$\lim_{\varepsilon \rightarrow 0} \widehat{F}'_\varepsilon(x) = F'(x). \quad (4.7)$$

This provides us with another stability result. The derivative of $F(x)$ is continuous with respect to perturbation of L by $\widehat{\sigma}(\varepsilon)B$. As a curiosity, we can do the following: By independence of $L(1)$ and $B(1)$, we have

$$\begin{aligned} \mathbb{E} \left[f(x + \widehat{L}_\varepsilon(1)) \frac{B(1)}{\widehat{\sigma}(\varepsilon)} \right] &= \mathbb{E} \left[\left(f(x + \widehat{L}_\varepsilon(1)) - f(x + L(1)) \right) \frac{B(1)}{\widehat{\sigma}(\varepsilon)} \right] \\ &\quad + \mathbb{E} \left[f(x + L(1)) \frac{B(1)}{\widehat{\sigma}(\varepsilon)} \right] \\ &= \mathbb{E} \left[B^2(1) \frac{f(x + \widehat{\sigma}(\varepsilon)B(1) + L(1)) - f(x + L(1))}{\widehat{\sigma}(\varepsilon)B(1)} \right]. \end{aligned}$$

Notice, that the fraction on the right is close to a Malliavin derivative, since we are looking at a derivative of f along B . Loosely speaking, when taking the limit we are looking at a derivative of $f(x + L(1))$ in the direction of $B(1)$, which resembles the idea of Malliavin differentiation. Furthermore, it is expected that this limit will be independent of $B(1)$, which has variance equal to 1. Informally, we have therefore given a link between the Malliavin derivative based on Brownian motion and the delta for Lévy processes. Hence, this motivates that the approach by Davis and Johansson [24] may be extended to more general Lévy processes than merely Brownian motion and Poisson processes as is the case in their paper. The formalization of this procedure is studied in Chapter 5.

Let us consider an example where the smoothing of L may be an attractive procedure. The so-called CGMY distribution was suggested in Carr *et al.* [20] to model asset price returns. It does not have any explicit density function, but is defined through its cumulant function¹

$$\psi_{\text{CGMY}}(\theta) = C\Gamma(-Y) \{ (M - i\theta)^Y + (G - i\theta)^Y - G^Y \}, \quad (4.8)$$

with $\Gamma(x)$ being the Gamma-function and constants C, G, M and Y . We suppose that C, G and M are positive, and $Y \in [0, 2)$. The CGMY distribution is infinitely divisible, and we can define a CGMY-Lévy process L with Lévy measure

$$\ell(dz) = C|z|^{-1-Y} \exp(-(G1(z < 0) + M1(z > 0))|z|) dz. \quad (4.9)$$

Since we do not have explicitly the density function of L , the density method can not be used for deriving sensitivity estimates $F'(x)$. Instead we can apply the density method on the Brownian motion after smoothing L . We first verify the condition of Thm. 2.0.3 for the CGMY-Lévy process, showing that a smooth density indeed exists.

We have

$$\sigma^2(\varepsilon) = C \int_{|z| < \varepsilon} |z|^{1-Y} \exp(-(G1(z < 0) + M1(z > 0))|z|) dz,$$

and thus it is sufficient to verify the condition in Thm. 2.0.3 for $z > 0$. Note that $e^{-Mz} \geq e^{-M\varepsilon}$ for $0 \leq z \leq \varepsilon$, and it follows that

$$\int_0^\varepsilon z^{1-Y} e^{-Mz} dz \geq \frac{1}{2-Y} \varepsilon^{2-Y} e^{-M\varepsilon}.$$

¹The cumulant function is here the logarithm of the characteristic function

Let $\gamma = 2 - Y$ in Thm. 2.0.3, and we see that

$$\liminf_{\varepsilon \downarrow 0} \frac{\sigma^2(\varepsilon)}{\varepsilon^\gamma} \geq \frac{1}{2 - Y} > 0,$$

as long as $Y > 0$. Hence, there exists a smooth density for the CGMY-distribution when $Y \in (0, 2)$ and, if we would have this available, we could calculate $F'(x)$ via its logarithmic derivative. By smoothing we can approximate $F'(x)$ by

$$\widehat{F}'_\varepsilon(x) = \mathbb{E} \left[f(x + \widehat{L}_\varepsilon(1)) \frac{B(1)}{\sigma(\varepsilon)} \right].$$

The sensitivity weight will have a large variance for small ε , but it provides us with an expression that can be calculated using Monte Carlo simulations based on sampling of the CGMY-distribution and an independent normal distribution.

As an application, we consider an example from insurance. Let the loss of an insurance company be described by L , and x being the premium charged by the company to accept this risk. The question for the insurance company is to find a level x such that the net loss $x + L(1)$ is acceptable. A simple measure could be that the insurance company can only bear losses which are above a certain threshold, K say. Given a premium x , they want to calculate the probability of falling below the threshold K , which can be expressed by $P(x + L(1) < K)$. We find

$$P(x + L(1) < K) = \mathbb{E} [1_{\{x+L(1)<K\}}],$$

which therefore is an expectation functional on the form we have analyzed in this paper with $f(z) = 1_{\{z < K\}}$. Consider the derivative of this probability with respect to x , which we call the *marginal premium rate*:

$$F'(x) = \frac{d}{dx} \mathbb{E} [1_{\{x+L(1)<K\}}].$$

The marginal premium rate tells us how sensitive the loss probability is with respect to the premium. Of course, if we know the density of $L(1)$, $p_{L(1)}$, and this is differentiable, the marginal premium rate is straightforwardly calculated to be

$$F'(x) = -p_{L(1)}(K - x).$$

Thus, changing the premium by dx leads to a change in the loss probability of $-p_{L(1)}(K - x) dx$. However, if now the density of $L(1)$ is not known as is the case for the CGMY-distribution, we can not perform this simple calculation. By smoothing L , we find the approximation

$$\widehat{F}'_\varepsilon(x) = \mathbb{E} \left[1_{\{x+\widehat{L}_\varepsilon(1)<K\}} \frac{B(1)}{\widehat{\sigma}(\varepsilon)} \right].$$

Computations using conditional expectation lead to

$$\widehat{F}'_\varepsilon(x) = \mathbb{E} \left[\mathbb{E} \left[1_{\{x+\widehat{\sigma}(\varepsilon)B(1)+L(1)<K\}} \frac{B(1)}{\widehat{\sigma}(\varepsilon)} \mid L(1) \right] \right]$$

$$= -\mathbb{E} \left[p_{\widehat{\sigma}(\varepsilon)B(1)}(K - L(1) - x) \right],$$

with $p_{\widehat{\sigma}(\varepsilon)B(1)}$ being the density function of $\widehat{\sigma}(\varepsilon)B(1)$. Thus, also the approximation can be expressed as a density evaluated in $K - x$, however, in this case we need to take the expectation over $L(1)$. Furthermore, the density is singular when going to the limit.

Using the theory of distribution functions, we give a direct argument for the convergence of $\widehat{F}'_\varepsilon(x)$ to $F'(x)$. In fact by integration-by-parts, we have

$$\widehat{F}'_{\varepsilon,B}(x) = - (p_{\widehat{\sigma}(\varepsilon)B(1)}(K - x - \cdot), p_{L(1)})_2 = - (p_{\widehat{\sigma}(\varepsilon)B(1)}, p_{L(1)}(K - x - \cdot))_2$$

where $(\cdot, \cdot)_2$ is the inner product in $L^2(\mathbb{R})$, the space of square-integrable functions on \mathbb{R} . Since $p_{\widehat{\sigma}(\varepsilon)B(1)} \rightarrow \delta_0$ when $\widehat{\sigma}(\varepsilon) \rightarrow 0$, we find

$$\lim_{\varepsilon \rightarrow 0} \widehat{F}'_{\varepsilon,B}(x) = - (\delta_0, p_{L(1)}(K - x - \cdot))_2 = -p_{L(1)}(K - x) = F'(x).$$

This procedure may be carried through rigorously by using Schwartz distribution theory.

4.2.2 Some numerical issues

In the last part of this Section we turn our attention to some numerical issues concerning the use of the conditional density method for the above approximations.

For numerical purposes, it is of interest to know the rate of convergence of $F'_\varepsilon(x)$ to $F'(x)$. We have the following convergence speed for $F'_{\varepsilon,B,W}(x)$:

Proposition 4.2.5. *Suppose $b > 0$, L having finite variance and f being a Lipschitz continuous function. Then there exists a constant C depending on x , b , the Lipschitz constant of f , and the variance of $L(1)$ such that*

$$|F'_{\varepsilon,B,W}(x) - F'(x)| \leq C\sigma(\varepsilon).$$

Proof. From the triangle and Cauchy-Schwarz inequalities, we have

$$\begin{aligned} & |F'_{\varepsilon,B,W}(x) - F'(x)| \\ & \leq \mathbb{E} \left[|f(x + L_\varepsilon(1)) - f(x + L(1))| \frac{bW(1) + \sigma(\varepsilon)B(1)}{b^2 + \sigma^2(\varepsilon)} \right] \\ & \quad + \mathbb{E} \left[|f(x + L(1))| \left| \frac{bW(1) + \sigma(\varepsilon)B(1)}{b^2 + \sigma^2(\varepsilon)} - \frac{W(1)}{b} \right| \right] \\ & \leq \mathbb{E} [|f(x + L_\varepsilon(1)) - f(x + L(1))|^2]^{1/2} \frac{\mathbb{E} [(bW(1) + \sigma(\varepsilon)B(1))^2]^{1/2}}{b^2 + \sigma^2(\varepsilon)} \\ & \quad + \mathbb{E} [f^2(x + L(1))]^{1/2} \mathbb{E} \left[\left| \frac{bW(1) + \sigma(\varepsilon)B(1)}{b^2 + \sigma^2(\varepsilon)} - \frac{W(1)}{b} \right|^2 \right]^{1/2}. \end{aligned}$$

Letting K being the Lipschitz constant, we get

$$|F'_{\varepsilon,B,W}(x) - F'(x)| \leq \frac{K}{\sqrt{b^2 + \sigma^2(\varepsilon)}} \mathbb{E} [|L_\varepsilon(1) - L(1)|^2]^{1/2} + K \mathbb{E} [(x + L(1))^2]^{1/2}$$

$$\times \mathbb{E} \left[\left| \left(\frac{b}{b^2 + \sigma^2(\varepsilon)} - \frac{1}{b} \right) W(1) + \frac{\sigma(\varepsilon)}{b^2 + \sigma^2(\varepsilon)} B(1) \right|^2 \right]^{1/2}.$$

Since $W(1)$ and $B(1)$ are independent, we find the last expectation to be (after taking the square-root)

$$\sigma(\varepsilon)/b\sqrt{(b^2 + \sigma^2(\varepsilon))}.$$

Moreover,

$$L_\varepsilon(1) - L(1) = \sigma(\varepsilon)B(1) + \tilde{Z}_\varepsilon(1) - \lim_{\varepsilon \downarrow 0} \tilde{Z}_\varepsilon(1).$$

Note that the difference between $\tilde{Z}_\varepsilon(1)$ and $\lim_{\varepsilon \downarrow 0} \tilde{Z}_\varepsilon(1)$ is the jumps between 0 and ε . Due to independence of the jumps and the Brownian motion B , we get

$$\mathbb{E} [|L_\varepsilon(1) - L(1)|^2] = 2\sigma^2(\varepsilon).$$

Hence, the result follows. \square

We note that with minor modifications of the above proof we can show that

$$|F'_{\varepsilon,B}(x) - F'(x)| \leq C\sigma(\varepsilon),$$

where C is a positive constant (not necessarily equal to the constant in the proposition above). To show this result, we can simply let $b = 0$ in the proof and modify accordingly. Finally, it holds true for $\hat{F}'_\varepsilon(x)$ as well by similar arguments.

In practice, one uses Monte Carlo methods in order to calculate $F'_\varepsilon(x)$. We consider the case $F'_{\varepsilon,B}(x)$, and recall that the estimated value of this based on N Monte Carlo simulations is

$$F'_{\varepsilon,B}(x) \approx \sum_{n=1}^N f(x + l_{\varepsilon,n}) \frac{b_n}{\sigma(\varepsilon)},$$

where b_n and $l_{\varepsilon,n}$ are independent random draws of $B(1)$ and $L_\varepsilon(1)$, respectively. Note that in order to draw from $L_\varepsilon(1)$, we use the draw from $B(1)$. The Monte Carlo error (or rather the standard deviation of the error) is given by

$$\text{std}(f(x + L_\varepsilon(1))B(1)) / (\sqrt{N}\sigma(\varepsilon)).$$

Assume now for technical simplicity that f is bounded. Then, from dominated convergence and independence of L and B , we find

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \text{Var} \left[f(x + L_\varepsilon(1))B(1) \right] &= \lim_{\varepsilon \rightarrow 0} \left\{ \mathbb{E} \left[f^2(x + L_\varepsilon(1))B^2(1) \right] - \mathbb{E} \left[f(x + L_\varepsilon(1))B(1) \right]^2 \right\} \\ &= \mathbb{E} \left[f^2(x + L(1)) \right]. \end{aligned}$$

Hence,

$$\lim_{\varepsilon \rightarrow 0} \text{Var} \left(f(x + L_\varepsilon(1)) \frac{B(1)}{\sigma(\varepsilon)} \right) = \infty.$$

From this we can conclude the following. If we decide to use the conditional density method on pure-jump Lévy processes after first doing an approximation, the expression

to simulate will have a large variance for small ε . Indeed, when ε tends to zero the variance explodes. This means that for close approximations of $x + L(1)$ we will have an expression to simulate which has a very high variance, and therefore we need a very high number of samples to get a confident estimate of the delta. In conclusion, the method may become very inefficient and unstable, and variance-reducing techniques are called for in order to get reliable estimates.

4.3 Numerical examples

We consider some examples to illustrate the conditional density method and our findings on approximations.

Let us assume that L is an NIG-Lévy process, that is, a Lévy process with NIG-distributed increments. Supposing $L(1)$ being NIG distributed with parameters α , β , δ , and μ , the density is (see Barndorff-Nielsen [14])

$$p_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x - \mu)^2}\right)}{\sqrt{\delta^2 + (x - \mu)^2}}. \quad (4.10)$$

Here, K_λ is the modified Bessel function of the second order with parameter λ , which can be represented by the integral

$$K_\nu(z) = \frac{\sqrt{\pi} z^\nu}{2^\nu \Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} dt,$$

for $\nu > -\frac{1}{2}$ and $z > 0$. We apply the density method to find a sensitivity weight π . A direct differentiation gives

$$-\partial \ln p_{\text{NIG}}(x) = -\beta + \frac{x - \mu}{\delta^2 + (x - \mu)^2} \left\{ 1 - \frac{\alpha\sqrt{\delta^2 + (x - \mu)^2} K_1'(\alpha\sqrt{\delta^2 + (x - \mu)^2})}{K_1(\alpha\sqrt{\delta^2 + (x - \mu)^2})} \right\}.$$

We can now use the recursive relation for the derivative of the Bessel function K_λ , saying (see e.g. Rydberg [57])

$$K_1'(x) = -\frac{1}{2}K_0(x) - \frac{1}{2}K_2(x).$$

Using the recursion $K_2(z) = K_0(z) + (2/z)K_1(z)$ we reach

$$K_1'(z) = -\frac{1}{z}K_1(z) - K_0(z).$$

Inserting this into the expression of $-\partial \ln p_{\text{NIG}}$ yields,

$$\begin{aligned} \pi &= -\partial \ln p_{\text{NIG}}(L(1)) \\ &= -\beta + \frac{L(1) - \mu}{\delta^2 + (L(1) - \mu)^2} \left\{ 2 + \alpha\sqrt{\delta^2 + (L(1) - \mu)^2} \frac{K_0(\alpha\sqrt{\delta^2 + (L(1) - \mu)^2})}{K_1(\alpha\sqrt{\delta^2 + (L(1) - \mu)^2})} \right\}. \end{aligned} \quad (4.11)$$

Since this is a function of $L(1)$, it will be a variance optimal weight.

In applying the Monte Carlo simulation technique, it may be rather cumbersome to calculate the two modified Bessel functions K_1 and K_2 in order to calculate an outcome of the sensitivity weight π . In fact, for each draw we must perform such a calculation, which makes the method very inefficient due to the heavy computational burden involved in calculating Bessel functions. An alternative will then be to use an approximation, like for instance considering the smoothed random variable $\widehat{L}_\varepsilon(1)$ defined in (4.5). Using the conditional density argument, we find that the delta can be calculated by the expectation operator

$$\widehat{F}'_\varepsilon(x) = \mathbb{E} \left[f(x + L(1) + \varepsilon B(1)) \frac{B(1)}{\varepsilon} \right].$$

Hence, rather than doing numerical calculation of Bessel functions, we simulate from a normal distribution. From the analysis in this paper, letting $\varepsilon \rightarrow 0$ brings us back to the derivative we are interested in. Hence, for small ε 's, $\widehat{F}'_\varepsilon(x)$ should be reasonably close to $F'(x)$. We have tested this numerically in the following examples.

Let $\alpha = 50$, $\beta = \mu = 0$ and $\delta = 0.015$. These figures are not unreasonable estimates for the logreturns of a stock price on a daily scale, see Rydberg [57]. Further, we consider a function f being the payoff from a call option with strike $K = 100$, that is,

$$f(x) = \max(0, \exp(x) - 100).$$

We implemented to density method in Matlab by sampling a NIG-distribution using the technique in Rydberg [57] and calculating the Bessel functions $K_0(z)$ and $K_1(z)$ using the built-in Matlab function `besselk`. The approximation $\widehat{F}'_\varepsilon(x)$ was calculated by drawing samples from a standard normal distribution.

In Figure 4.1 we show the resulting derivatives for $x = \ln(S(0))$ with $S(0) = 100$ and $\varepsilon = 0.01$. Along the horizontal axis we have the number of samples (in 10^5) used in the estimation of the expectation operator, and the two expressions are calculated using common random numbers. The density method is depicted with a broken line, and we see that it has slightly less variance than the approximated derivative. But looking at the scale on the vertical axis, the approximation is pretty good, although it seems that it is slightly overestimating the true derivative. By reducing ε , we observe a convergence towards $F'(x)$, however, at the expense of a higher variance in the estimation of the expectation in $\widehat{F}'_\varepsilon(x)$. This is shown in Figure 4.2, where we plot the estimates as function of samples for three different values of ε , $\varepsilon = 0.01, 0.005$ and $\varepsilon = 0.001$. The smaller ε , the higher variance, which leads to a higher number of samples for ensuring accuracy of the estimate.

We note that our numerical example covers the delta of an at-the-money call option on a stock, where the delta is calculated one time step (one day, say) prior to the exercise date of the option. We get the delta by dividing the derivatives $F'(x)$ and $\widehat{F}'_\varepsilon(x)$ by $S(0) = 100$, resulting from an application of the chain rule.

To test our method on discontinuous functions f , we considered the above set-up for a digital option, that is, a payoff function $f(x) = 1(e^x > K)$ for some positive threshold K . The simulations showed that the derivative $\widehat{F}'_\varepsilon(x)$ had a significantly higher variance when estimated by Monte Carlo simulations. In fact, one needed to choose number of

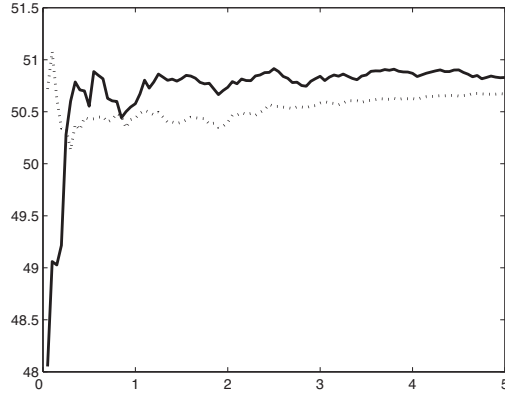


Figure 4.1: The estimated derivative based on the density method (broken line) versus an approximation using an added Brownian motion (solid line) as a function of the number of samples (in 10^5).

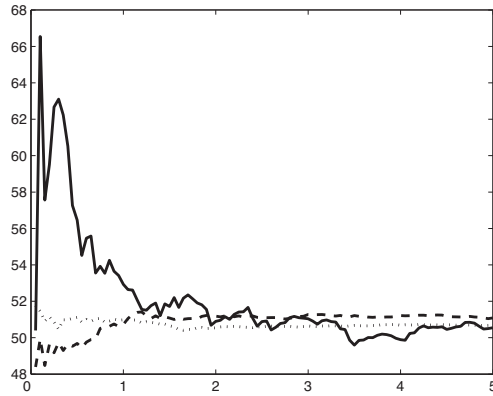


Figure 4.2: The estimated derivative based on an approximation using an added Brownian motion as a function of the number of samples (in 10^5). $\varepsilon = 0.01$ in dotted line, $\varepsilon = 0.005$ in broken line and $\varepsilon = 0.001$ in solid line.

samples several scales above what was required for the call option in order to get reasonable estimates for the approximated derivative. Hence, from a numerical perspective, discontinuous functions f seem to behave badly under approximations by Brownian motions when one applies it to pure-jump Lévy processes. Variance reducing techniques like *quasi*-Monte Carlo simulations may be fruitful and speed up the convergence in such situations.

As a final note in this numerical subsection, let us briefly discuss the issues concerning approximating the small jumps of a Lévy process by a Brownian motion. Following the idea in this paper, the small jumps are approximated by $\sigma(\varepsilon)B(t)$ for a suitable scaling $\sigma(\varepsilon)$. The sensitivity weight is of the form

$$\pi = \frac{B(1)}{\sigma(\varepsilon)},$$

if we have no continuous martingale part in the Lévy process and decide to use the density method with respect to the Brownian motion B . In order to simulate $F'_{\varepsilon,B}(x)$, we must sample from $B(1)$ and $L_\varepsilon(1)$. The latter is equivalent to sample from a compound Poisson process since we have only jumps of size bigger than ε . Indeed, we must sample from a compound Poisson process with jump size distribution given by

$$\mathbf{1}_{|z| \geq \varepsilon} \ell(dz)/c,$$

where the normalizing constant c is defined as

$$c = \ell(|z| \geq \varepsilon).$$

The jump intensity will be c . This is in principle simple to simulate as long as one has a routine to sample for the truncated Lévy measure and knows the constant c . However, using for instance Markov Chain Monte Carlo methods, one can sample from the jump distribution without knowing the constant c .

4.4 Conclusion

In this study we have considered the problem of robustness of the sensitivity parameter delta to model choice. Our models are selected within the Lévy family, but they differ according to how the presence of small jumps is taken into account.

First, following the study in Asmussen and Rosinski [3], we have considered models with small jumps, see L in (2.1) and their approximations given by models of type L_ε (2.3), where a continuous martingale part with controlled standard deviation is replacing the small jumps. In this case both models have the same total variance and $L_\varepsilon(t) \rightarrow L(t)$, for $\varepsilon \downarrow 0$. Secondly, we have considered a smoothing \widehat{L}_ε of the Lévy process L . Also in this case we have $\widetilde{L}_\varepsilon(t) \rightarrow L(t)$, for $\varepsilon \downarrow 0$, but there is no control on the variances between the two models. The two situations can be usefully applied in different contexts.

In both cases we have addressed the question of the robustness of the parameter delta

$$\begin{aligned} F'(x) &= \frac{d}{dx} E[f(x + L(t))] \\ F'_\varepsilon(x) &= \frac{d}{dx} E[f(x + L_\varepsilon(t))] \\ \widehat{F}'_\varepsilon(x) &= \frac{d}{dx} E[f(x + \widehat{L}_\varepsilon(t))]. \end{aligned}$$

We have applied different methods of computation: the classical density method and the newly introduced conditional density method. The different computational techniques for the delta lead to different weights. However the values of the sensitivity are the same. Qualitatively, the conditional density method is an application of computations similar to the ones in the density method, but applied after having performed some conditioning (this inspired by the Malliavin method à la Davis and Johansson [24]). In our analysis we have considered functions f with different degrees of regularity, always keeping in mind the needs coming from applications to finance and insurance. Our examples include also the digital option.

Indeed a robustness result is proved, i.e.

$$\begin{aligned} F'_\varepsilon(x) &\longrightarrow F'(x), \quad \varepsilon \downarrow 0 \\ \widehat{F}'_\varepsilon(x) &\longrightarrow F'(x), \quad \varepsilon \downarrow 0. \end{aligned}$$

If this is reassuring when coming to applications, we also remark that we experience some curious situations important from the numerical point of view. In fact, according to the different methods applied, some representations of the deltas turn out to be highly inefficient. This is evident when we consider models L with no original continuous martingale component (i.e. $b = 0$ in (2.1)) and we take the corresponding L_ε as approximating model. In this case the conditional density method shows an exploding variance of the random variable that must be simulated. This yielding to the need of a large number of samples to get some confident estimate of the delta.

Robustness of option prices and their deltas in markets modeled by jump-diffusions

This chapter is extracted from the paper "Robustness of option prices and their deltas in markets modeled by jump-diffusions" by Fred Espen Benth, Giulia Di Nunno, and Asma Khedher, available at *E-print*, No. 2, January(2010), Department of Mathematics, University of Oslo, Norway. To appear in *Comm. Stoch. Analysis*.

We study the robustness of option prices to model variation within a jump-diffusion framework. In particular we consider models in which the small variations in price dynamics are modeled with a Poisson random measure with infinite activity and models in which these small variations are modeled with a Brownian motion. We show that option prices are robust. Moreover we study the computation of the deltas in this framework with two approaches, the Malliavin method and the Fourier method. We show robustness of the deltas to the model variation.

This chapter is organized as follows. In Section 5.1 we give a short introduction to the Malliavin calculus for mixtures of Gaussian and compensated Poisson random measures. Section 5.2 is dedicated to jumps-diffusions and results about the robustness of the models and the option prices. Section 5.3 deals directly with the computation of the deltas and related analysis of robustness to the model. Here both the Malliavin and the Fourier approaches are introduced.

5.1 Chaotic representation for Lévy processes

In Itô [47], multiple stochastic integrals with respect to a Poisson random measure are defined (see Di Nunno [25] for an extension to general random measures with independent values). We recall the construction, which follows the same steps as in the Wiener case (see Kuo [48]).

Here and in the sequel we assume that the Lévy measure satisfies

$$\sigma^2(\infty) := \int_{\mathbb{R}_0} z^2 \ell(dz) < \infty. \quad (5.1)$$

Consider a Lévy process L having a representation as in (2.1) with $b = 1$. Introduce the measure M on the Borel σ -algebra $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that for $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$,

$$M(E) = \int_{E(0)} dt + \int_{E'} z^2 dt \ell(dz),$$

where $E(0) = \{t \in \mathbb{R}_+; (t, 0) \in E\}$ and $E' = E - \{(t, 0) \in E\}$. Define

$$\mu(E) = \int_{E(0)} dW(t) + \lim_{n \rightarrow \infty} \int_{\{(t,z) \in E; \frac{1}{n} < |z| < n\}} z \tilde{N}(dt, dz),$$

where the convergence is in $L^2(\Omega)$. We denote by $L^2(\Omega) := L^2(\Omega, \mathcal{F}, \mathbb{P})$ the Hilbert space of square-integrable random variables, equipped with the norm $\|F\|_2 = (\mathbb{E}[F^2])^{1/2} < \infty$. The set function μ is a centered random measure such that for $E_1, E_2 \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $M(E_1) < \infty$ and $M(E_2) < \infty$,

$$\mathbb{E}[\mu(E_1)\mu(E_2)] = M(E_1 \cap E_2).$$

Denote by $L_n^2 = L^2((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}((\mathbb{R}_+ \times \mathbb{R})^n), M^{\otimes n})$, with the standard norm $|\cdot|_n$. Let

$$f = \mathbf{1}_{E_1 \times \dots \times E_n},$$

where the sets $E_1, \dots, E_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ are pairwise disjoint and

$$M(E_1) < \infty, \dots, M(E_n) < \infty.$$

The multiple stochastic integral of the elementary function f is an element in $L^2(\Omega)$ defined as follows

$$I_n(f) := \int_{(\mathbb{R}_+ \times \mathbb{R})^n} f d\mu^{\otimes n} := \mu(E_1) \cdots \mu(E_n).$$

By standard arguments, I_n can be extended to the symmetric function in L_n^2 by appealing to linearity and continuity. Moreover, for any symmetric functions $f \in L_n^2$ and $g \in L_m^2$ we have

$$\mathbb{E}[I_n(f)I_m(g)] = \delta_{n,m} n! \int_{(\mathbb{R}_+ \times \mathbb{R})^n} \tilde{f} \tilde{g} d\mu^{\otimes n},$$

where $\delta_{n,m} = 1$, if $n = m$ and 0 otherwise and $\tilde{\cdot}$ denotes the symmetrization of a given function. Itô [47] proves the following chaos expansion for elements of $L^2(\Omega)$:

Theorem 5.1.1. *For any $F \in L^2(\Omega)$ there exists a unique sequence $(f_n)_{n=0}^\infty$ of symmetric functions $f_n \in L_n^2$ such that*

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

(with convergence in $L^2(\Omega)$). Moreover, it holds

$$\|F\|_2^2 = \sum_{n=0}^{\infty} n! |f_n|_n^2.$$

Note that, among all the stochastic measures with independent values in $L^2(\Omega)$ it is only in the case of mixtures of Gaussian and Poisson measures that it is possible to achieve chaos representation type of results. This is proved in Theorem 2.2 in Di Nunno [25].

In Solé, Utzet and Vives [65] (see also Di Nunno [25] for random measures with independent values) a stochastic derivative is defined on a subspace of $L^2(\Omega)$. The idea is to exploit chaos expansion representations much in the same manner as done for the Malliavin derivative in the Wiener space (see Nualart [54]). Suppose $F \in L^2(\Omega)$ has a chaotic representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ such that

$$\sum_{n=1}^{\infty} nn! |f_n|_n^2 < \infty. \quad (5.2)$$

Then, the *Malliavin derivative* $DF : \mathbb{R}_+ \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ of F is the random field defined as

$$D_{\zeta} F := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\zeta, \cdot)), \quad \zeta \in \mathbb{R}_+ \times \mathbb{R}, \quad (5.3)$$

with convergence in $L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, M \otimes \mathbb{P})$. Note that the Malliavin derivative can be viewed as an annihilation operator, shifting the chaos expansion of F by one to the left.

Denote by $\text{Dom} D$ the set of functionals $F \in L^2(\Omega)$ that satisfy (5.2). This becomes a Hilbert space equipped with the scalar product

$$\langle F, G \rangle = \mathbb{E}[FG] + \mathbb{E}\left[\int_{\mathbb{R}_+ \times \mathbb{R}} D_{\zeta} F D_{\zeta} G M(d\zeta)\right],$$

on which D is a closed operator from $\text{Dom} D$ to $L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, M \otimes \mathbb{P})$. Furthermore, let $\text{Dom} D^0$ be the set of random variables $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} nn! \int_{\mathbb{R}_+ \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}} f_n^2((t, 0), \zeta_1, \dots, \zeta_{n-1}) dt dM^{\otimes(n-1)}(\zeta_1, \dots, \zeta_{n-1}) < \infty,$$

For $F \in \text{Dom} D^0$ we define the square integrable stochastic process

$$D_{t,0} F := \sum_{n=1}^{\infty} n I_{n-1}(f_n((t, 0), \cdot)),$$

where the convergence is in $L^2(\mathbb{R}_+ \times \Omega, dt \otimes \mathbb{P})$. Analogously, for $\ell(dz) \neq 0$, let $\text{Dom} D^J$ be the set of $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} nn! \int_{(\mathbb{R}_+ \times \mathbb{R}_0) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}} f_n^2((t, z), \zeta_1, \dots, \zeta_{n-1}) dM^{\otimes(n-1)}(\zeta_1, \dots, \zeta_{n-1}) < \infty.$$

For $F \in \text{Dom} D^J$, define the random field $D_{t,z}^J F : \mathbb{R}_+ \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ such that

$$D_{t,z}^J F := \sum_{n=1}^{\infty} n I_{n-1}(f_n((t, z), \cdot)),$$

where the convergence is in $L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega, z^2 dt d\ell(x) \otimes \mathbb{P})$. We remark that the derivative $D_{t,0}$ is essentially a derivative with respect to the Brownian part of L , and in many situations the usual rules of classical Malliavin calculus on Wiener space apply.

Let $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ and $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ be the canonical spaces for the Brownian motion and pure jump Lévy process, resp. We can interpret

$$\Omega = \Omega_W \times \Omega_J, \quad \mathcal{F} = \mathcal{F}_W \otimes \mathcal{F}_J, \quad \mathbb{P} = \mathbb{P}_W \otimes \mathbb{P}_J.$$

The following chain rule for $D_{t,0}$ is proved by Solé, Utzet and Vives [65].

Proposition 5.1.1. *Assume $F = f(Z, Z') \in L^2(\Omega_W \times \Omega_J)$, with $Z \in \text{Dom} D^W$, $Z' \in L^2(\Omega_J)$, and $f(x, y)$ being a continuously differentiable function with bounded partial derivative in the first variable. Then $F \in \text{Dom} D^0$, and*

$$D_{t,0}F = \frac{\partial f}{\partial x}(Z, Z') D_t^W Z,$$

where D^W is the Malliavin derivative in $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ and $\text{Dom} D^W$ its domain.

In Solé, Utzet and Vives [65] the Skorohod integral with respect to a mixture of Gaussian and Poisson random measures is also defined (see Di Nunno [25] and Di Nunno and Rozanov [27] for the treatment with respect to general stochastic measures in $L^2(\Omega)$). Let us consider

$$G(\zeta) = \sum_{n=0}^{\infty} I_n(\widehat{f}_n(\zeta, \cdot)), \quad \zeta \in \mathbb{R}_+ \times \mathbb{R},$$

where $f_n \in L_{n+1}^2$ is symmetric in the last n variables. We denote \widehat{f}_n the symmetrization of f_n in all $n+1$ variables. If

$$\sum_{n=0}^{\infty} (n+1)! |\widehat{f}_n|_{n+1}^2 < \infty, \tag{5.4}$$

the Skorohod integral of $G(\zeta)$, $\zeta \in \mathbb{R}_+ \times \mathbb{R}$, is defined by

$$\delta(G) := \sum_{n=0}^{\infty} I_{n+1}(\widehat{f}_n),$$

where the convergence of the series on the right-hand side is in $L^2(\Omega)$. Denote by $\text{Dom} \delta$ the set of random fields $G(\zeta)$ satisfying (5.4). The following is a duality formula proved by Solé, Utzet and Vives [65]:

Proposition 5.1.2. *Let $G \in L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mu \otimes \mathbb{P})$. The random field G belongs to $\text{Dom} \delta$ if and only if there is a constant C such that for all $F \in \text{Dom} D$,*

$$|\mathbb{E}[\int_{\mathbb{R}_+ \times \mathbb{R}} G(\zeta) D_\zeta F M(d\zeta)]| \leq C \|F\|_2.$$

If $G \in \text{Dom} \delta$, then $\delta(G)$ is the element of $L^2(\Omega)$ characterized by

$$\mathbb{E}[\delta(G)F] = \mathbb{E}[\int_{\mathbb{R}_+ \times \mathbb{R}} G(\zeta) D_\zeta F M(d\zeta)],$$

for any $F \in \text{Dom} D$.

The Malliavin derivatives introduced above will become useful when we analyze the delta of option prices based on jump-diffusion models, see Section 5.3 .

5.2 Robustness of jump-diffusions and option prices

In this Section we consider the robustness of jump-diffusions given by the solution of stochastic differential equations of the form

$$\begin{aligned} X(t) = x &+ \int_0^t \alpha(X(s-)) ds + \int_0^t \beta(X(s-)) dW(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} \gamma(X(s-), z) \tilde{N}(ds, dz). \end{aligned} \quad (5.5)$$

We assume that the coefficient functions $\alpha(x)$ and $\beta(x)$ have linear growth and are Lipschitz continuous and that γ is of the form $\gamma(x, z) = \gamma_1(x)g(z)$, $x \in \mathbb{R}$, $z \in \mathbb{R}_0$, where the (stochastic) factor $\gamma_1(x)$ has linear growth and is Lipschitz continuous and the (deterministic) factor $g(z)$ satisfies

$$G^2(\infty) = \int_{\mathbb{R}_0} g^2(z) \ell(dz) < \infty,$$

which will ensure that $X(t)$ has finite variance. We also define

$$G^2(\varepsilon) = \int_{|z| < \varepsilon} g^2(z) \ell(dz),$$

for later use. Notice that $G^2(\varepsilon)$ converges to zero when $\varepsilon \downarrow 0$. A jump-diffusion of type (5.5) is, e.g., considered in Example 5.3.

Note that we consider a stochastic differential equation with the roles of W and \tilde{N} separated, that is, we do not consider an equation using L as the integrator, but rather split the roles of the continuous martingale and the pure-jump parts. This is more in line with common formulations of such stochastic differential equations (see for example Davis and Johansson [24]). Introduce the approximating jump-diffusion dynamics where the small jumps part in (5.5) has been substituted by a Brownian motion B independent of W and appropriately scaled, namely

$$\begin{aligned} X_\varepsilon(t) = x &+ \int_0^t \alpha(X_\varepsilon(s-)) ds + \int_0^t \beta(X_\varepsilon(s-)) dW(s) \\ &+ \int_0^t \left(\int_{|z| < \varepsilon} (\gamma^2(X_\varepsilon(s-), z) \ell(dz))^{1/2} dB(s) \right. \\ &+ \left. \int_0^t \int_{|z| \geq \varepsilon} \gamma(X_\varepsilon(s-), z) \tilde{N}(ds, dz) \right) \\ &= x + \int_0^t \alpha(X_\varepsilon(s-)) ds + \int_0^t \beta(X_\varepsilon(s-)) dW(s) \\ &+ \int_0^t G(\varepsilon) \gamma_1(X_\varepsilon(s-)) dB(s) \\ &+ \int_0^t \int_{|z| \geq \varepsilon} \gamma(X_\varepsilon(s-), z) \tilde{N}(ds, dz). \end{aligned} \quad (5.6)$$

The existence and uniqueness of the solutions $X(t)$ and $X_\varepsilon(t)$ are ensured by the following theorem collected from Ikeda and Watanabe [46] (Thm 9.1. Chap IV):

Theorem 5.2.1. *Let U be an open set in \mathbb{R}_0 , α and β be two measurable functions $\mathbb{R} \rightarrow \mathbb{R}$ and γ be a measurable function $\mathbb{R} \times U \rightarrow \mathbb{R}$ such that, for some positive constant K ,*

$$|\alpha(x)|^2 + |\beta(x)|^2 + \int_U |\gamma(x, z)|^2 \ell(dz) \leq K(1 + |x|^2), \quad x \in \mathbb{R}, \quad (5.7)$$

$$|\alpha(x) - \alpha(y)|^2 + |\beta(x) - \beta(y)|^2 + \int_U |\gamma(x, z) - \gamma(y, z)|^2 \ell(dz) \leq K|x - y|^2, \quad x, y \in \mathbb{R}. \quad (5.8)$$

Then there exists a unique \mathcal{F}_t -adapted right-continuous process $X(t)$ with left-hand limits which satisfies the following stochastic differential equation

$$\begin{aligned} X(t) = x + \int_0^t \alpha(X(s-)) ds + \int_0^t \beta(X(s-)) dW(s) \\ + \int_0^t \int_U \gamma(X(s-), z) \tilde{N}(ds, dz). \end{aligned} \quad (5.9)$$

Before proving that $X_\varepsilon(t)$ converges to $X(t)$ in $L^2(\Omega)$, we need a lemma which shows the boundedness of X in $L^2([0, T] \times \Omega)$ for $T < \infty$.

Lemma 5.2.1. *Let $X(t)$ and $X_\varepsilon(t)$, $t \in [0, T]$, be the unique solutions of (5.5) and (5.6), respectively. For every $0 \leq t \leq T < \infty$, we have the following type of estimate for the respective norms*

$$\|X(t)\|_2^2, \|X_\varepsilon(t)\|_2^2 \leq ae^{bt},$$

where a and b are positive constants depending on T but independent of ε in the case of X_ε .

Proof. By the Cauchy-Schwartz inequality and the application of the Itô isometry, we find that

$$\begin{aligned} \|X(t)\|_2^2 \leq C|x|^2 + CTE \left[\int_0^t \alpha^2(X(s)) ds \right] + CE \left[\int_0^t \beta^2(X(s)) ds \right] \\ + CG^2(\infty) \mathbb{E} \left[\int_0^t \gamma_1^2(X(s)) ds \right], \end{aligned}$$

for some positive constant C . By linear growth, it follows that $|\alpha(x)|^2 \leq K(1 + |x|^2)$ for some positive constant K . Hence, by using the same property for β and γ_1 , it follows that

$$\|X(t)\|_2^2 \leq C_1 + C_2 \int_0^t \|X(s)\|_2^2 ds,$$

for two positive constants C_1, C_2 , which depend only on $K, T, G^2(\infty)$ and x . By Gronwall's inequality, the lemma follows for $X(t)$.

Concerning the estimate for $X_\varepsilon(t)$, we proceed in the way as for $X(t)$. In this case, however, we get an additional contribution from the term

$$\int_0^t G(\varepsilon) \gamma_1(X_\varepsilon(s)) dB(s),$$

whereas the jump-term is including only jumps in absolute value greater than ε . However, after applying the Itô isometry, we can merge the contributions from these two terms into $G^2(\infty)\mathbb{E}[\int_0^t \gamma_1^2(X_\varepsilon(s)) ds]$. Hence, we are back to the same estimation type as for $X(t)$. This completes the proof. \square

We use the lemma to prove the following robustness result:

Proposition 5.2.1. *For every $0 \leq t \leq T < \infty$, we have*

$$\|X(t) - X_\varepsilon(t)\|_2^2 \leq CG^2(\varepsilon),$$

where X and X_ε are solutions of (5.5) and (5.6), respectively and C is a positive constant depending on T , but independent of ε .

Proof. We have

$$\begin{aligned} X(t) - X_\varepsilon(t) = & \int_0^t (\alpha(X(s-)) - \alpha(X_\varepsilon(s-))) ds \\ & + \int_0^t (\beta(X(s-)) - \beta(X_\varepsilon(s-))) dW(s) \\ & + \int_0^t \int_{0 < |z| < \varepsilon} \gamma(X(s-), z) \tilde{N}(ds, dz) \\ & - \int_0^t G(\varepsilon) \gamma_1(X(s-)) dB(s) \\ & + \int_0^t G(\varepsilon) (\gamma_1(X(s-)) - \gamma_1(X_\varepsilon(s-))) dB(s) \\ & + \int_0^t \int_{|z| \geq \varepsilon} (\gamma(X(s-), z) - \gamma(X_\varepsilon(s-), z)) \tilde{N}(ds, dz). \end{aligned}$$

Therefore, using the Hölder inequality and the Itô isometry, we get

$$\begin{aligned} \|X(t) - X_\varepsilon(t)\|_2^2 \leq & T\mathbb{E} \left[\int_0^t (\alpha(X(s)) - \alpha(X_\varepsilon(s)))^2 ds \right] \\ & + \mathbb{E} \left[\int_0^t (\beta(X(s)) - \beta(X_\varepsilon(s)))^2 ds \right] \\ & + 2G^2(\varepsilon)\mathbb{E} \left[\int_0^t \gamma_1^2(X(s)) ds \right] \\ & + G^2(\varepsilon)\mathbb{E} \left[\int_0^t (\gamma_1(X(s)) - \gamma_1(X_\varepsilon(s)))^2 ds \right] \\ & + \left(G^2(\infty) - G^2(\varepsilon) \right) \mathbb{E} \left[\int_0^t (\gamma_1(X(s)) - \gamma_1(X_\varepsilon(s)))^2 ds \right]. \end{aligned}$$

Hence, by the Lipschitz continuity of the three coefficient functions and the triangle inequality, we find

$$\|X(t) - X_\varepsilon(t)\|_2^2 \leq K(T + 1 + G^2(\infty)) \int_0^t \|X(s) - X_\varepsilon(s)\|_2^2 ds$$

$$+ 2G^2(\varepsilon)K \int_0^t (1 + \|X(s)\|_2^2) ds.$$

Applying Gronwall's inequality and Lemma 5.2.1, we prove the Proposition. \square

This result has various applications, one of which is the numerical simulations of the solution of (5.5). First, we observe that the speed of convergence is explicitly given by $G(\varepsilon)$, which in many situations will be a rate of ε . See e.g. Asmussen and Rosinski [3] for examples in the case $g(z) = z$. In practice, it may be difficult to simulate from a Lévy process L directly. One may in such circumstances approximate the small jumps by an appropriate scaled Brownian motion and observe that the remaining process is a compound Poisson process. Brownian motion and compound Poisson processes are simple to simulate on a computer, and the approximating dynamics may next be discretized for instance, by an Euler scheme. Our result in Prop. 5.2.1 provides the mathematical foundation for such a procedure, ensuring for instance that expectation functionals of the type $\mathbb{E}[f(X_\varepsilon(t))]$ converge to $\mathbb{E}[f(X(t))]$ under mild assumptions on f . We have the following corollary:

Corollary 5.2.1. *Suppose f is a Lipschitz continuous function and X and X_ε solve (5.5) and (5.6), resp. Then, for every $0 \leq t \leq T < \infty$, there exists a positive constant C depending on T but independent of ε such that*

$$|\mathbb{E}[f(X_\varepsilon(t))] - \mathbb{E}[f(X(t))]| \leq CG(\varepsilon).$$

Proof. Letting K be the Lipschitz constant of f , we have from the Jensen inequality,

$$|\mathbb{E}[f(X_\varepsilon(t))] - \mathbb{E}[f(X(t))]| \leq K\mathbb{E}[|X_\varepsilon(t) - X(t)|].$$

Hence, from the Cauchy-Schwarz inequality and Prop. 5.2.1 the result follows. \square

This result has an immediate interpretation in terms of robustness of option prices. If we assume that $X(t)$ represents the dynamics of some asset on which there is written an option with payoff $f(X(t))$ at an exercise time t , then the discounted risk-neutral expected value of $f(X(t))$ is the option price. Supposing that we model $X(t)$ directly under the risk-neutral probability (i.e., assuming \mathbb{P} is the risk-neutral probability), the discounted asset dynamics must be a martingale, that is, $\alpha(x) = rx$, with r being the risk-free interest rate. But the approximating dynamics X_ε is also a martingale after discounting when $\alpha(x) = rx$, and henceforth, we obtain from the Corollary above that option prices are stable with respect to perturbation in the underlying dynamics when we substitute small jumps with an appropriate continuous martingale. In practical terms, we may interpret this as having two competing models, one where we suppose that small variations in the asset dynamics come from a jump process of infinite activity, and another where we model this by continuous martingale. It is very hard, if possible, to decide which model is better from a statistical point of view. However, the result above shows that the effect on option prices is very small. From a different perspective, if we want to perform a numerical evaluation of the option price, we may apply the above result in order to quantify the error if we approximate small jumps by a Brownian motion dynamics. The error is explicit in terms of $G(\varepsilon)$, the volatility of the jumps smaller than ε .

5.3 Computation of the Delta using the Malliavin method and robustness

In this section we present the Malliavin approach to compute the delta for option prices based on a jump-diffusion market model. Our approach extends the method proposed in Davis and Johansson [24]. We apply the results to study robustness of the delta to small-jump approximations in the underlying jump-diffusion model. These results explain to us that we may use the Malliavin approach to approximate the delta in cases when there is no continuous martingale part in the jump-diffusion dynamics.

Let $\mathcal{F}_t^{\tilde{N}} = \sigma\{\int_0^s \int_A \tilde{N}(du, dz); \quad s \leq t, \quad A \in \mathcal{B}(R_0)\}$. Assume that α , β and γ are continuously differentiable functions with bounded derivatives and consider Markov jump diffusions, X of the form (5.5), for which we have a continuously differentiable function h with bounded derivative in the first argument such that

$$X(t) = h(X^c(t), X^d(t)), \quad X(0) = x. \quad (5.10)$$

Here X^c satisfies a stochastic differential equation

$$\begin{aligned} dX^c(t) &= \alpha_c(X^c(t))dt + \beta_c(X^c(t))dW(t), \\ X^c(0) &= x = h(X^c(0), X^d(0)), \end{aligned} \quad (5.11)$$

with continuously differentiable coefficients α_c , β_c , while X^d is adapted to the natural filtration $\mathcal{F}^{\tilde{N}}$ of the compensated compound Poisson process \tilde{N} . In particular, X^d does not depend on x . The jump-diffusion process of type (5.10) is called *separable*.

We associate with the process X^c , a process V given by

$$V(t) = 1 + \int_0^t \alpha'_c(X^c(s))V(s)ds + \int_0^t \beta'_c(X^c(s))V(s)dW(s), \quad (5.12)$$

The process V is called *the first variation process* for X^c and we have

$$V(t) = \frac{\partial X^c(t)}{\partial x}.$$

Theorem 5.3.1. *Let X be a diffusion of the form (5.5). We assume that it is separable. Define*

$$\Gamma = \left\{ a \in L^2[0, T] \mid \int_0^T a(t)dt = 1 \right\}.$$

Then for $a \in \Gamma$ and $f(X(T)) \in L^2(\Omega)$,

$$\Delta = \mathbb{E}\left[f(X(T)) \int_0^T a(t)\beta_c^{-1}(X^c(t))V(t)dW(t)\right], \quad (5.13)$$

where V is given by (5.12).

Proof. Assume that $f \in C_K^\infty(\mathbb{R})$. Then

$$\frac{\partial}{\partial x} \mathbb{E}\left[f(X(T))\right] = \mathbb{E}\left[f'(X(T)) \frac{\partial X(T)}{\partial x}\right] = \mathbb{E}\left[f'(X(T)) \frac{\partial X(T)}{\partial X^c(T)} V(T)\right], \quad (5.14)$$

where V is the first variation process for X^c . By the chain rule (Proposition 5.1.1), we have

$$D_{t,0}X(T) = \frac{\partial X(T)}{\partial X^c(T)} D_t^W X^c(T) = \frac{\partial X(T)}{\partial X^c(T)} V(T)(V(t))^{-1} \beta_c(X^c(t)).$$

See Proposition 2.2.1 for more details. Therefore,

$$\frac{\partial X(T)}{\partial X^c(T)} V(T) = D_{t,0}X(T) V(t) \beta_c^{-1}(X^c(t)).$$

Multiply by $a(t)$ and integrate,

$$\frac{\partial X(T)}{\partial X^c(T)} V(T) = \int_0^T D_{t,0}X(T) a(t) \beta_c^{-1}(X^c(t)) V(t) dt. \quad (5.15)$$

Inserting (5.15) in (5.14), the chain rule (Proposition 5.1.1) and the Duality formula (Proposition 5.1.2) yield

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E}[f(X(T))] &= \mathbb{E} \left[\int_0^T f'(X(T)) D_{t,0}X(T) a(t) \beta_c^{-1}(X^c(t)) V(t) dt \right] \\ &= \mathbb{E} \left[\int_0^T D_{t,0}f(X(T)) a(t) \beta_c^{-1}(X^c(t)) V(t) dt \right] \\ &= \mathbb{E} \left[f(X(T)) \int_0^T a(t) \beta_c^{-1}(X^c(t)) V(t) dW(t) \right]. \end{aligned}$$

Then we can extend this formula to $f(X(T)) \in L^2(\Omega)$ following the Proposition 2.2.2. \square

We provide an example of a jump-diffusion dynamics satisfying our assumptions and at the same time illustrating the result (5.13).

Example

Consider a jump-diffusion of the form

$$dX(t) = \alpha X(t-)dt + \beta X(t-)dW(t) + \int_{\mathbb{R}_0} (e^z - 1)X(t-) \tilde{N}(ds, dz), \quad (5.16)$$

where α and β are constants. We introduce the process $X^c(t)$ defined by

$$\begin{aligned} dX^c(t) &= \left\{ \alpha + \int_{\mathbb{R}_0} (1 + z - e^z) \ell(dz) \right\} X^c(t) dt + \beta X^c(t) dW(t), \\ X(0) &= x. \end{aligned}$$

Then by applying the Itô formula to $\hat{X}(t) = e^{\tilde{Z}(t)} X^c(t)$, where

$$\tilde{Z}(t) = \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(dt, dz),$$

we get,

$$\begin{aligned}
d\widehat{X}(t) &= e^{\widetilde{Z}(t-)} dX^c(t) + \int_{\mathbb{R}_0} (e^{\widetilde{Z}(t-)+z} X^c(t) - e^{\widetilde{Z}(t-)} X^c(t)) \widetilde{N}(dt, dz) \\
&+ X^c(t) e^{\widetilde{Z}(t-)} \int_{\mathbb{R}_0} (-1 - z + e^z) \ell(dz) dt \\
&= \alpha \widehat{X}(t-) dt + \beta \widehat{X}(t-) dW(t) + \int_{\mathbb{R}_0} (e^z - 1) \widehat{X}(t-) \widetilde{N}(dt, dz), \quad (5.17)
\end{aligned}$$

Therefore, $\widehat{X}(t) = X(t)$, a.e. and we see that the process X given by equation (5.16) is a separable process. Now, to illustrate the result in Theorem 5.3.1, we consider a differentiable claim $f(X(T)) = X^2(T)$, where X is given by (5.16) with $\alpha = 0$ and $\beta = 1$. In this case, an explicit solution of X is given by $X(t) = \exp\{W(t) - \frac{t}{2} + \int_0^t \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz)\}$ and the first variation process is $V(t) = X^c(t)/x$. We can apply the formula (5.13) with $a(t) = 1/T$ and easily see that

$$\Delta = \frac{1}{xT} \mathbb{E}[X^2(T)W(T)] = \frac{xe^{-T}}{T} \mathbb{E}\left[W(T)e^{2W(T)}e^{2\int_0^T \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz)}\right].$$

Put $Y(T) = e^{2\int_0^T \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz)}$. Since the two random variables $W(T)$ and $Y(T)$ are independent, we have

$$\Delta = \frac{xe^{-T}}{T} \mathbb{E}[W(T)e^{2W(T)}] \mathbb{E}[Y(T)] = 2xe^T \mathbb{E}[Y(T)].$$

On the other hand side note that in this example the delta can be computed directly by simple differentiation, this gives

$$\begin{aligned}
\Delta &= 2xe^{-T} \mathbb{E}[e^{2W(T)}e^{2\int_0^T \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz)}] \\
&= 2xe^{-T} \mathbb{E}[e^{2W(T)}] \mathbb{E}[Y(T)] = 2xe^T \mathbb{E}[Y(T)].
\end{aligned}$$

This confirms the result found before.

Let X_ε be a jump diffusion of the form (5.6). We assume that it is separable. Then the process X_ε^c is given by

$$X_\varepsilon^c(t) = x + \int_0^t \alpha_c(X_\varepsilon^c(s)) ds + \int_0^t \beta_c(X_\varepsilon^c(s)) dW(s) + \int_0^t G(\varepsilon) \gamma_{1,c}(X_\varepsilon^c(s)) dB(s)$$

and the first variation process V_ε of X_ε^c is given by

$$\begin{aligned}
V_\varepsilon(t) &= 1 + \int_0^t \alpha'_c(X_\varepsilon^c(s)) V_\varepsilon(s) ds + \int_0^t \beta'_c(X_\varepsilon^c(s)) V_\varepsilon(s) dW(s) \\
&+ \int_0^t G(\varepsilon) \gamma'_{1,c}(X_\varepsilon^c(s)) V_\varepsilon(s) dB(s).
\end{aligned}$$

We are now ready to study the delta related to the approximating model. We propose four ways of applying the Malliavin approach with related assumptions. The first two

(5.18) and (5.19) are completely equivalent in the sense that the computations can be carried out either with respect to the original Brownian component W or with respect to the additional one B . The expression (5.20) derived from the fact that the evaluation of the delta depends on the distribution and we consider a Brownian motion $\widetilde{W}_\varepsilon$ that merges W and B . In the last case, (5.21), the delta is computed starting from an approximating model created by modifying the coefficients of the original Brownian component W instead of considering a new independent Brownian motion B .

Theorem 5.3.2. *Let X_ε be a diffusion of the form (5.6) and assume that it is separable. Let $a \in \Gamma$, V_ε the first variation process of X_ε^c and $f(X_\varepsilon(T)) \in L^2(\Omega)$. Then*

$$\Delta_\varepsilon = \mathbb{E} \left[f(X_\varepsilon(T)) \int_0^T a(t) \beta_c^{-1}(X_\varepsilon^c(t)) V_\varepsilon(t) dW(t) \right], \quad (5.18)$$

$$\Delta_\varepsilon = \mathbb{E} \left[f(X_\varepsilon(T)) \int_0^T a(t) \gamma_{1,c}^{-1}(X_\varepsilon^c(t)) \frac{V_\varepsilon(t)}{G(\varepsilon)} dB(t) \right]. \quad (5.19)$$

We assume $\beta(x) = \gamma_1(x)$. Then

$$\Delta_\varepsilon = \mathbb{E} \left[f(X_\varepsilon(T)) \int_0^T a(t) \gamma_{1,c}^{-1}(X_\varepsilon^c(t)) \frac{V_\varepsilon(t)}{\sqrt{G^2(\varepsilon) + 1}} d\widetilde{W}_\varepsilon(t) \right], \quad (5.20)$$

where $\widetilde{W}_\varepsilon(t) = \frac{1}{\sqrt{G^2(\varepsilon)+1}}W(t) + \frac{G(\varepsilon)}{\sqrt{G^2(\varepsilon)+1}}B(t)$.

If we approximate the small jumps of $X(t)$ (equation (5.5)) by $X_\varepsilon(t)$, where $B(t) = W(t)$, then

$$\Delta_\varepsilon = \mathbb{E} \left[f(X_\varepsilon(T)) \int_0^T a(t) \{G(\varepsilon) \gamma_{1,c}(X_\varepsilon^c(t)) + \beta_c(X_\varepsilon^c(t))\}^{-1} V_\varepsilon(t) dW(t) \right]. \quad (5.21)$$

Proof. By the chain rule (Proposition 5.1.1), we have

$$D_{t,0} X_\varepsilon(T) = \frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)} D_t^W X_\varepsilon^c(T).$$

Here, D^W is the Malliavin derivative with respect to the Brownian motion W . By Thm 2.2.1 in Nualart [54],

$$\begin{aligned} D_t^W X_\varepsilon^c(T) &= \beta_c(X_\varepsilon^c(t)) + \int_t^T \alpha'_c(X_\varepsilon^c(s)) D_t^W X_\varepsilon^c(s) ds \\ &+ \int_t^T \beta'_c(X_\varepsilon^c(s)) D_t^W X_\varepsilon^c(s) dW(s) \\ &+ \int_t^T G(\varepsilon) \gamma'_{1,c}(X_\varepsilon^c(s)) D_t^W X_\varepsilon^c(s) dB(s). \end{aligned}$$

Then

$$D_t^W X_\varepsilon^c(T) = V_\varepsilon(T) (V_\varepsilon(t))^{-1} \beta_c(X_\varepsilon^c(t)).$$

However, we find the expression (5.18) for the Δ_ε following the same steps of the Thm 5.3.1.

We can apply the chain rule again with differentiation taken with respect to B (Proposition 5.1.1), then we get

$$D_{t,0}X_\varepsilon(T) = \frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)} D_t^B X_\varepsilon^c(T),$$

where D^B is the Malliavin derivative with respect to the Brownian motion B . Then, following the same steps as above we obtain the expression (5.19) for the Δ_ε .

We assume now that we are in the case of the approximation (5.6), with $\beta(x) = \gamma_1(x)$. Then the process X_ε^c is given by

$$X_\varepsilon^c(t) = x + \int_0^t \alpha_c(X_\varepsilon^c(s)) ds + \int_0^t \gamma_{1,c}(X_\varepsilon^c(s)) \sqrt{G^2(\varepsilon) + 1} d\widetilde{W}_\varepsilon(t).$$

By Thm 5.3.1, expression (5.20) follows. The last case (5.21) also follows by application of Thm 5.3.1. \square

Note that, if $\varepsilon = 0$, we are in the case of no-approximation and we have the same method as proposed in Davis and Johansson [24], except for more general jump parts. This shows us how to use the Malliavin approach for these jump diffusions of general type. Next, in the case of jump-diffusions with no continuous component, i.e. $\beta = 0$, we have an expression which can be used as the approximation for the delta.

We next address the question of robustness of the delta with respect to approximations of the small jumps by an appropriately scaled continuous martingale. It turns out that this question can be efficiently answered by means of *Fourier transform*. The methods of Fourier transform will translate the question of convergence of the delta to a question of convergence of the derivative of the characteristic function of the approximating dynamics. One may ask why we do not study the expression derived above for the delta directly. The reason is that in the singular case of $\beta = 0$, the expressions inside the expectation for the delta in Thm 5.3.2 will involve singular weights which in general are hard to study in the limit (see Chapter 4 for simple examples of such singular weights). The Fourier approach avoids this problem.

The approach we choose can be used also for efficient computations of the delta, however, only for those cases where the characteristic function is easily computable which is in general not the case for stochastic differential equations like (5.5) and (5.6). We also note that the application of the Fourier transform requires also the explicit solution of the first variation process dynamics (5.24).

Assume that $f, \hat{f} \in L^1(\mathbb{R})$. From the inverse Fourier transform of f (equation (3.16)), we have

$$\begin{aligned} \mathbb{E}[f(X_\varepsilon^x(t))] &= \int_{\mathbb{R}} \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iyu} \hat{f}(u) du \right\} P_{X_\varepsilon^x(t)}(dy) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} e^{-iuy} P_{X_\varepsilon^x(t)}(dy) \right\} \hat{f}(u) du \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(u) \mathbb{E} [e^{-iuX_\varepsilon^x(t)}] du, \end{aligned} \tag{5.22}$$

where $P_{X_\varepsilon^x(t)}(dy)$ is the distribution of $X_\varepsilon(t) = X_\varepsilon^x(t)$, the solution of (5.6) with $X_\varepsilon(0) = X_\varepsilon^x(0) = x$. Fubini-Tonelli's Theorem (see Folland [33]) is applied to commute the integrations. Similarly, we get for $X(t) = X^x(t)$ being the solution of (5.5) with $X(0) = X^x(0) = x$,

$$\mathbb{E}[f(X^x(t))] = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(u) \mathbb{E}[e^{-iuX^x(t)}] du. \quad (5.23)$$

Thus, in order to study the delta, we need to be able to move differentiation inside the inverse Fourier transform. But, furthermore, we must have accessible the derivative of $X_\varepsilon^x(t)$ and $X^x(t)$ with respect to x . Before moving on with the robustness of deltas, we study this.

Introduce the stochastic differential equation

$$\begin{aligned} Y^y(t) = y &+ \int_0^t \alpha'(X^x(s-)) Y^y(s-) ds + \int_0^t \beta'(X^x(s-)) Y^y(s-) dW(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} \gamma'(X^x(s-), z) Y^y(s-) \tilde{N}(ds, dz). \end{aligned} \quad (5.24)$$

Since the derivatives of α, β and γ are assumed to be bounded, it follows from Thm. 5.2.1 that there exists a unique solution $Y^y(t)$ of (5.24). From Thm 40 in Chapter V of Protter [56], it follows that $X^x(t)$ is differentiable with respect to x , and that

$$\frac{\partial X^x(t)}{\partial x} = Y^1(t) \quad (\text{i.e. } y = 1). \quad (5.25)$$

By the same considerations, $X_\varepsilon^x(t)$ is differentiable with respect to x , and

$$\frac{\partial X_\varepsilon^x(t)}{\partial x} = Y_\varepsilon^1(t), \quad (5.26)$$

with $Y_\varepsilon^y(t)$ being the unique solution of the stochastic differential equation

$$\begin{aligned} Y_\varepsilon^y(t) = y &+ \int_0^t \alpha'(X_\varepsilon^x(s-)) Y_\varepsilon^y(s-) ds + \int_0^t \beta'(X_\varepsilon^x(s-)) Y_\varepsilon^y(s-) dW(s) \\ &+ \int_0^t G(\varepsilon) \gamma'_1(X_\varepsilon^x(s-)) Y_\varepsilon^y(s-) dB(s) \end{aligned} \quad (5.27)$$

$$+ \int_0^t \int_{|z| \geq \varepsilon} \gamma'(X_\varepsilon^x(s-), z) Y_\varepsilon^y(s-) \tilde{N}(ds, dz). \quad (5.28)$$

We have the following regularity of Y and Y_ε :

Proposition 5.3.1. *Let $Y^y(t)$ and $Y_\varepsilon^y(t)$ be the solutions of (5.24) and (5.27), resp. For $0 \leq t \leq T < \infty$ it holds that*

$$\|Y^y(t)\|_2^2, \|Y_\varepsilon^y(t)\|_2^2 < ae^{bt},$$

for positive constants a and b depending on T but independent of ε in the case of Y_ε . Moreover,

$$\|Y^y(t) - Y_\varepsilon^y(t)\|_2^2 \leq CG^2(\varepsilon),$$

for a positive constant C independent of ε .

Proof. The proof follows the same lines as the arguments for Lemma 5.2.1 and Prop. 5.2.1. The only modification is that we use the boundedness of the derivatives $\alpha'(x), \beta'(x)$ and $\gamma'(x)$ rather than the Lipschitz continuity of α, β and γ . \square

In the next Proposition we derive the expressions for the delta based on X and X_ε using the Fourier method.

Proposition 5.3.2. *Let $X^x(t)$ and $Y^y(t)$ be solutions of (5.5) and (5.24), resp., and $X_\varepsilon^x(t)$ and $Y_\varepsilon^y(t)$ of (5.6) and (5.27), resp. Let $u\hat{f}(u) \in L^1(\mathbb{R})$. Then, for $0 \leq t \leq T$,*

$$\begin{aligned}\frac{\partial}{\partial x} \mathbb{E}[f(X^x(t))] &= \frac{1}{2\pi} \int_{\mathbb{R}} (-iu) \hat{f}(u) \mathbb{E}[Y^1(t) e^{-iuX^x(t)}] du \\ \frac{\partial}{\partial x} \mathbb{E}[f(X_\varepsilon^x(t))] &= \frac{1}{2\pi} \int_{\mathbb{R}} (-iu) \hat{f}(u) \mathbb{E}[Y_\varepsilon^1(t) e^{-iuX_\varepsilon^x(t)}] du.\end{aligned}$$

Proof. First, by dominated convergence (or appropriate result in Folland [33], Proposition 2.27), we can move the differentiation inside the integral and inside the expectation operator on the right-hand side in (5.23). Next, differentiating, we obtain straightforwardly the results since $Y^1(t) = \partial X^x(t)/\partial x$. We follow exactly the same argument for $X_\varepsilon^x(t)$. This proves the result. \square

Finally, we state our result on robustness:

Proposition 5.3.3. *Let $u\hat{f}(u) \in L^1(\mathbb{R})$. For $0 \leq t \leq T$, it holds that*

$$\lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial x} \mathbb{E}[f(X_\varepsilon^x(t))] = \frac{\partial}{\partial x} \mathbb{E}[f(X^x(t))].$$

Proof. Cauchy-Schwarz gives:

$$\begin{aligned}& |\mathbb{E}[Y_\varepsilon^1(t) e^{-iuX_\varepsilon^x(t)} - Y^1(t) e^{-iuX^x(t)}]| \\ & \leq \mathbb{E}[|Y_\varepsilon^1(t) - Y^1(t)|] + \mathbb{E}[|Y^1(t)| |e^{-iuX_\varepsilon^x(t)} - e^{-iuX^x(t)}|] \\ & \leq \mathbb{E}[|Y_\varepsilon^1(t) - Y^1(t)|^2]^{1/2} + \mathbb{E}[|Y^1(t)|^2]^{1/2} \mathbb{E}[|e^{-iuX_\varepsilon^x(t)} - e^{-iuX^x(t)}|^2]^{1/2} \\ & \leq CG^2(\varepsilon) + \tilde{C} \mathbb{E}[|e^{-iuX_\varepsilon^x(t)} - e^{-iuX^x(t)}|^2]^{1/2}\end{aligned}$$

In the last estimation, we have used Prop. 5.3.1 where C, \tilde{C} are two positive constants independent of ε . Moreover, the function $\exp(-iux)$ is Lipschitz continuous, which is seen from the polar coordinate representation, and thus the final term is also majorised by a constant times $G(\varepsilon)$ by Prop 5.2.1. Hence,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[Y_\varepsilon^1(t) e^{-iuX_\varepsilon^x(t)}] = \mathbb{E}[Y^1(t) e^{-iuX^x(t)}].$$

By appealing to Prop. 5.3.1 again, we see that $\mathbb{E}[Y_\varepsilon^1(t) \exp(-iuX_\varepsilon^x(t))]$ can be bounded uniformly in ε , and hence by dominated convergence the Proposition follows. \square

Note that the above results applying the Fourier method hold also for the case $\beta = 0$. In particular, this tells that even in the singular case, i.e. when the process $X(t)$ does not have any continuous martingale part, the delta for the approximating option price based on $X_\varepsilon^x(t)$ and calculated based on Malliavin differentiation with respect to the Brownian component will converge to the true value.

We remark that there is no requirement of continuity of f in the above arguments. In Chapter 4, Section 4.2, we gave examples of functions f satisfying the integrability restrictions.

Computation of Greeks in multi-factor models with applications to power and commodity markets

This chapter is extracted from the paper "Computation of Greeks in multi-factor models with applications to power and commodity markets", available at *E-print*, No. 5, March (2010), Department of Mathematics, University of Oslo, Norway, submitted for publication.

We study the computation of the Greeks of options written on assets modelled by a multi-factor dynamics. For this purpose, we apply the conditional density method in which the knowledge of the density of one factor is enough to derive expressions for the Greeks not involving any differentiation of the payoff function. Several examples are given in applications to power and commodity markets, including numerical examples.

The chapter is organized as follows. In the next section, we present a general multi-factor spot model which is a typical model describing commodities and electricity spot prices. In Sect. 6.2 we discuss the computation of the Greeks delta and gamma using the conditional density method and we give several examples in which the knowledge of the density of just one factor of the multi-factor model is needed. In Sect. 6.3 we present a discussion on options on forwards and we compute the delta using the conditional density method. In Sect. 6.4 we consider several numerical examples illustrating the applicability of our approach. Finally, we conclude in Sect. 6.5

6.1 Multi-factor models in commodity and power markets

Let us consider a probability space (Ω, \mathcal{F}, Q) equipped with a filtration \mathcal{F}_t , $0 \leq t \leq T$, where T is some finite time horizon. Suppose that the spot price dynamics $S(t)$, $0 \leq t \leq T$, is defined by

$$S(t) = g(t, X_1(t), \dots, X_n(t)), \quad (6.1)$$

for n independent adapted stochastic processes $X_1(t), \dots, X_n(t)$ representing the factors. The function $g : \mathbb{R}^n \mapsto \mathbb{R}$ is continuous to ensure that $S(t)$ is adapted as well. Note that we work directly under the risk-neutral probability measure Q .

The spot price model in (6.1) is very general and encompasses many interesting cases known in the energy and commodity markets. We list here a few examples and connect them to our model.

The Schwartz model is defined as

$$S(t) = S(0) \exp(X(t)), \quad (6.2)$$

where

$$dX(t) = (\theta - \alpha X(t)) dt + \sigma dW(t), \quad (6.3)$$

for $\theta \in \mathbb{R}$, α, σ positive constants, and W a Brownian motion under Q . By letting $n = 1$, $X_1(t) = X(t)$ and $g(x) = S(0) \exp(x)$ we have identified the Schwartz model to (6.1). The Schwartz model has been applied as a simple model for the oil price dynamics in Schwartz [60]. In this model, the log-price mean-reverts towards a level given by the θ .

A two-factor extension is proposed by Schwartz and Smith [61] (see also Lucia and Schwartz [50]). It takes the form

$$S(t) = S(0) \exp(X(t) + Y(t)), \quad (6.4)$$

where $Y(t)$ is the long-term non-stationary drift of the price. It is given by

$$dY(t) = \mu dt + \eta d\widetilde{W}(t), \quad (6.5)$$

with \widetilde{W} a Brownian motion under Q , possibly correlated with W . In the case \widetilde{W} is independent of W , then we obviously choose $n = 2$, $X_1(t) = X(t)$, $X_2(t) = Y(t)$, and the function g is set equal to $g(x, y) = S(0) \exp(x + y)$. In the correlated case, we represent $Y(t)$ in the following way:

$$Y(t) = Y(0) + \mu t + \eta \rho W(t) + \eta \sqrt{1 - \rho^2} \widetilde{W}_1(t),$$

with ρ being the correlation between \widetilde{W} and W while \widetilde{W}_1 is independent of W . Then we can consider an $n = 3$ factor model with $X_1(t) = X(t)$, $X_2(t) = Y(0) + \mu t + \eta \rho W(t)$, and $X_3(t) = \eta \sqrt{1 - \rho^2} \widetilde{W}_1(t)$. The function g is naturally extended: $g(x, y, z) = S(0) \exp(x + y + z)$.

An extension with stochastic volatility of the Schwartz and Smith model is found in Geman [39]. One may choose

$$dX(t) = (\theta - \alpha X(t)) dt + \sqrt{Z(t)} dW(t), \quad (6.6)$$

where $Z(t)$ is some positive adapted stochastic process such that its square-root is Itô integrable. An example, applied by Geman [39], is to assume that Z follows the Heston model. In Benth [6], the Schwartz model is considered with the stochastic volatility Z following a superposition of non-Gaussian Ornstein-Uhlenbeck processes, as proposed by Barndorff-Nielsen and Shephard [5]. In the Schwartz-Smith model with stochastic volatility the elements g , X_1 , and X_2 are identified in a similar way, however, the process $X_1(t)$ becomes more complex.

A natural extension of the models above is to include jumps. In Benth, Saltyte Benth, and Koekebakker [11] a general class based on non-stationary jump processes is discussed.

We consider here some examples for illustration. In the power markets, spikes are frequently observed and one natural model for this is

$$S(t) = S(0) \exp(X(t) + Y(t)), \quad (6.7)$$

where

$$dY(t) = -\beta Y(t) dt + dL(t). \quad (6.8)$$

Here, L is a Lévy process, which may possibly be time-inhomogeneous (in this case, also called additive process) in order to model the seasonal jump frequency which is naturally observed in many power markets. The constant β is positive.

Another model with jumps is proposed by Cartea and Figueroa [17] and applied to UK power prices. The spot price is given by

$$S(t) = \exp(Y(t)), \quad (6.9)$$

where

$$dY(t) = -\alpha Y(t) dt + \sigma dW(t) + \ln J dq(t). \quad (6.10)$$

Here the random jump size J is lognormal, i.e. $\ln J$ follows a normal distribution with mean μ_J and variance σ_J^2 , and dq is a Poisson process such that

$$dq(t) = \begin{cases} 1, & \text{with probability } l dt \\ 0, & \text{with probability } 1 - l dt. \end{cases} \quad (6.11)$$

here l is the intensity of the process.

In general, realistic models should take into account the seasonality. The standard way to include seasonality in the type of models above is, for example, to let

$$S(t) = \Lambda(t) \exp(X(t)) \quad (\Lambda(t) > 0)$$

in the Schwartz model, where the deterministic $\Lambda(t)$ represents the seasonality component. We then have $S(0) = \Lambda(0) \exp(X(0))$ and $g(t, x) = \Lambda(t) \exp(x)$. Modifications of the other examples above to include seasonality are straightforward.

Benth, Kallsen, and Meyer-Brandis [9] proposed an additive model for the electricity spot price defined as a superposition of independent Ornstein-Uhlenbeck processes:

$$S(t) = \Lambda(t) \sum_{i=1}^n Y_i(t). \quad (6.12)$$

with

$$dY_i(t) = -\lambda_i Y_i(t) dt + dL_i(t). \quad (6.13)$$

Here, the constants λ_i are all positive and $L_i(t)$ are subordinator processes (i.e. increasing Lévy processes) possibly being time-inhomogeneous. By using subordinators as jump components, one is assured to have a spot price with positive values. The natural way to apply the model in practice (see *e.g.* Benth, Kiesel, and Nazarova [10]) is to separate the model into base components and one or more spike components. For instance, the

two first factors may account for the normal variations in the market, the so-called base signal, while a third component may model the spikes, i.e. big jumps followed by a fast mean-reversion. In many markets the jump frequency is seasonally varying, leading to a time-inhomogeneous subordinator for this factor. Thus, the distributional properties are not in general analytically available. We remark that in Meyer-Brandis and Tankov [51] it is proposed to model the base signal using a Brownian motion driven Ornstein-Uhlenbeck process. The identification of the model (6.12) to our general multi-factor dynamics in (6.1) is obvious. Note, however, that this identification is not unique as we see hereafter. In fact, from (6.13) we have

$$Y_i(t) = Y_i(0)e^{-\lambda_i t} + \int_0^t e^{-\lambda_i(t-s)} dL_i(s).$$

Thus, in the multi-factor representation, we can choose

$$X_i(t) = \int_0^t e^{-\lambda_i(t-s)} dL_i(s),$$

and

$$g(t, x_1, \dots, x_n) = \Lambda(t) \sum_{i=1}^n \{Y_i(0)e^{-\lambda_i t} + x_i\}. \quad (6.14)$$

This is the representation we shall use. In the following, when we are going to study Greeks for options, observe that the process $S(t)$, $0 \leq t \leq T$, could also be interpreted as some index, like an index on a stock exchange, or some value of a basket of assets. In this case, the factors $X_i(t)$ would represent the individual assets. This is also covered by our considerations in this present paper, although we particularly focus on models for commodities and power.

6.2 Options on spot prices and their Greeks

We consider European options written on the spot price $S(t)$, $0 \leq t \leq T$, with exercise time T and payoff function $h : \mathbb{R} \mapsto \mathbb{R}$. The arbitrage-free price is defined as

$$C(S(0)) = e^{-rT} \mathbb{E}[h(S(T))], \quad (6.15)$$

where we have emphasized the dependency on $S(0)$ since we are going to compute the Greeks with respect to this. The parameter r is the risk-free instantaneous interest rate of a bond used as numéraire. A standing assumption in the sequel is that $h(S(T)) \in L^1(Q)$ to make the price $C(S(0))$ well-defined. To this end, note that we can write

$$h(S(T)) = h(g(T, X_1(T), \dots, X_n(T))).$$

We suppose now that there exist a function f and a differentiable function ζ such that

$$h(S(T)) = f(X_1(T) + \zeta(S(0)), X_2(T), \dots, X_n(T)). \quad (6.16)$$

Hence, we consider prices

$$C(S(0)) = e^{-rT} \mathbb{E}[f(X_1(T) + \zeta(S(0)), X_2(T), \dots, X_n(T))]. \quad (6.17)$$

6.2.1 The delta

Denote by p_1 the density of $X_1(T)$, which we suppose to be known and let $\partial \ln p_1$ be its logarithmic derivative. In the next proposition we derive the delta of C .

Proposition 6.2.1. *Assume that there exists an integrable function u on \mathbb{R} such that*

$$|\mathbb{E}[f(x, X_2(T), \dots, X_n(T))]| p_1(x - \zeta(S(0))) \leq u(x). \quad (6.18)$$

Then

$$\frac{\partial C}{\partial S(0)} = -\zeta'(S(0))e^{-rT} \mathbb{E}[h(S(T))\partial \ln p_1(X_1(T))].$$

Proof. By conditioning on $X_1(T)$ we find

$$\begin{aligned} & \mathbb{E}[f(X_1(T) + \zeta(S(0)), X_2(T), \dots, X_n(T))] \\ &= \int_{\mathbb{R}} \mathbb{E}[f(x + \zeta(S(0)), X_2(T), \dots, X_n(T))] p_1(x) dx \\ &= \int_{\mathbb{R}} \mathbb{E}[f(x, X_2(T), \dots, X_n(T))] p_1(x - \zeta(S(0))) dx. \end{aligned}$$

With our assumption, appealing to Thm 2.27 in Folland [33], we can move the differentiation inside the integration, and find the result after dividing and multiplying by $p_1(x)$. Hence, the proof is complete. \square

Remarks

1. In the assumptions above, we have assumed that the density of $X_1(T)$ is defined on the real line, and implicitly that it is strictly positive there. We can easily adapt the result to densities only defined on the positive half-axis, see Example 6.4.2 in Sect. 6.4.
2. Note that we have assumed the knowledge of the density of $X_1(T)$. In practice, we search for the factor with the most convenient density, among those factors for which a density is known, and use this as the first factor.
3. From a computational point of view, the result above is highly advantageous. First of all, we do not need to differentiate explicitly the payoff function h (or equivalently, f), a procedure that is not always possible since the payoff may not be differentiable (e.g. digital options). Furthermore, by applying Monte Carlo methods in conjunction with moving the derivative into the payoff function, we would get a very slow convergence due to very high variability. On the contrary, using Monte Carlo to compute the expectation in the Prop 6.2.1 turns out to be much more stable. We shall demonstrate this in the numerical examples in Sect. 6.4.

The delta is essentially the price of a new option with payoff $h(S(T))\partial \ln p_1(X_1(T))$, namely, the option payoff h modified by the logarithmic derivative of the density of the first

factor. The delta takes thus a general form where *only* the payoff changes across options. In most of the examples we will look at, the factor X_1 will be chosen as the Gaussian process appearing in the model considered and therefore very simple to simulate using standard software. In fact, one may simulate the delta in parallel with the option price $C(S(0))$ in a Monte Carlo approach. We look at some examples.

Example

We start with considering the two-factor model of Schwartz and Smith for the case of independence between W and \widetilde{W} . We represent the payoff as

$$h(S(T)) = h(\exp(\ln(S(0)) + X_1(T) + X_2(T)))$$

and thus we find $f(x_1 + \zeta(S(0)), x_2) = h(\exp(\ln(S(0)) + x_1 + x_2))$ with $\zeta(s) = \ln s$. Both factors $X_1(T)$ and $X_2(T)$ are Gaussian random variables, with explicit mean and variance, and may be used as the first factor for the calculation of the delta. If $X_1(t)$ is the mean-reverting process, we have that the mean is $X(0) \exp(-\alpha t) + \theta(1 - \exp(-\alpha t))/\alpha$ and variance $\sigma^2(1 - \exp(-2\alpha t))/2\alpha$. Therefore, it holds that

$$\partial \ln p_1(x) = -\frac{1}{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha T})} \left(x - X(0)e^{-\alpha T} - \frac{\theta}{\alpha}(1 - e^{-\alpha T}) \right).$$

Hence, the delta is

$$\frac{\partial C}{\partial S(0)} = \frac{e^{-rT} 2\alpha}{S(0)\sigma^2(1 - e^{-2\alpha T})} \mathbb{E} \left[h(S(T)) \left(X_1(T) - X(0)e^{-\alpha T} - \frac{\theta}{\alpha}(1 - e^{-\alpha T}) \right) \right]. \quad (6.19)$$

It is simple to modify the above expression for the case of dependent factors. We see that the expression of the delta remains the same if the second factor is a mean-reverting jump process, mimicking spikes, as in (6.8). Hence, we do not see any different delta except of course for the change in the properties of the second factor. Note that when $X_2(t)$ is the Gaussian model, we may apply the density method directly, since $X_1(T) + X_2(T)$ is again a Gaussian random variable. The mean in this case is

$$e^{-\alpha_1 t} X_1(0) + \frac{\theta_1}{\alpha_1}(1 - e^{-\alpha_1 t}) + e^{-\alpha_2 t} X_2(0) + \frac{\theta_2}{\alpha_2}(1 - e^{-\alpha_2 t})$$

and variance (in the independent case) is

$$\frac{\sigma_1^2}{2\alpha_1}(1 - e^{-2\alpha_1 t}) + \frac{\sigma_2^2}{2\alpha_2}(1 - e^{-2\alpha_2 t}).$$

Thus, a simple application of the density method would give

$$\begin{aligned} \frac{\partial C}{\partial S(0)} = & \frac{2\alpha_1\alpha_2 e^{-rT}}{S(0)(\sigma_1^2\alpha_2(1 - e^{-2\alpha_1 T}) + \sigma_2^2\alpha_1(1 - e^{-2\alpha_2 T}))} \mathbb{E} \left[h(S(T)) \left(X_1(T) + X_2(T) \right. \right. \\ & \left. \left. - e^{-\alpha_1 T} X_1(0) - \frac{\theta_1}{\alpha_1}(1 - e^{-\alpha_1 T}) - e^{-\alpha_2 T} X_2(0) - \frac{\theta_2}{\alpha_2}(1 - e^{-\alpha_2 T}) \right) \right]. \end{aligned}$$

The direct application of the density method to the sum of the factors $X_1(T) + X_2(T)$ is not possible in the case of a jump process in the second component $X_2(T)$, except in the case when the distribution density of $X_2(t)$ is known. But in that case still assuming independence, the joint distribution of X_1 and X_2 is rather complicated, being the convolution of a Gaussian distribution with the distribution of X_2 . In fact the convolution may give a density which is not analytically tractable. This is a typical situation where the conditional density method provides an easy alternative.

Example

We may extend the Schwartz and Smith model to include a stochastic volatility. In this case it is natural to switch the roles of X_1 and X_2 , and let $X_1(t) = X_1(0) + \mu t + \eta \widetilde{W}(t)$. Thus, $X_1(T)$ is normally distributed with mean $X_1(0) + \mu T$ and variance $\eta^2 T$. The logarithmic derivative of p_1 is in this case

$$\partial \ln p_1(x) = -\frac{1}{\eta^2 T} (x - X(0) - \mu T) .$$

Note that ζ and f remain the same. Hence, again by assuming independence between the two factors for simplicity, we obtain the following expression for the delta

$$\frac{\partial C}{\partial S(0)} = \frac{e^{-rT}}{S(0)\eta^2 T} \mathbb{E} [h(S(T)) (X_1(T) - X(0) - \mu T)] .$$

We observe that this is an alternative expression for the delta in the constant-volatility case as well. In fact, the stochastic volatility only enters in the dynamics of $X_2(t)$ and it is nowhere appearing in the other terms involved in the delta. In this sense we see that the delta is “independent” of the structure of the stochastic volatility process.

Example

Consider the multi-factor model in (6.12). By using the representation in (6.14) with $X_i(t) = \int_0^t e^{-\lambda_i(t-s)} dL_i(s)$, we can write

$$h(S(T)) = h \left(S(0)e^{-\lambda_1 T} \frac{\Lambda(T)}{\Lambda(0)} - \Lambda(T) \sum_{i=2}^n (e^{-\lambda_1 T} - e^{-\lambda_i T}) Y_i(0) + \Lambda(T) \sum_{i=1}^n X_i(T) \right) ,$$

where we recognize that

$$\zeta(S(0)) = \frac{S(0)e^{-\lambda_1 T}}{\Lambda(0)} ,$$

and

$$f(x_1 + \zeta(S(0)), \dots, x_n) = h \left(\Lambda(T)(x_1 + \zeta(S(0)) - \Lambda(T)) \sum_{i=2}^n (e^{-\lambda_1 T} - e^{-\lambda_i T}) Y_i(0) + \Lambda(T) \sum_{i=2}^n x_i \right) .$$

So, we find that

$$\zeta'(S(0)) = \frac{e^{-\lambda_1 T}}{\Lambda(0)} .$$

The density p_1 of $X_1(T)$ is not necessarily simple to find explicitly in this model, although its cumulant can be calculated straightforwardly from the cumulant of the subordinator L_1 . However, a reasonable approximation of p_1 , when T is sufficiently large, is to apply its stationary distribution. This is frequently known, since it is specified in the modeling process and estimated to data. In the numerical examples in Sect. 6.4, we shall consider this alternative.

One may ask which payoff functions h satisfy the boundedness condition (6.18) on f in Prop. 6.2.1. If h is bounded, then obviously f will be, and then the boundedness condition will hold since the density p_1 is integrable. This will then include plain vanilla put and digital options, for example. With a view towards call options (and insisting on not applying the call-put parity), it is natural to consider functions h which have at most exponential linear growth (having the exponential models in mind). The boundedness assumption (6.18) will then look like: there exists a function u being integrable such that

$$K(1 + \exp(|x|))p_1(x) \leq u(x).$$

Here, the constant K involves the expectation of the exponential in the remaining factors. If p_1 is a Gaussian density, then $p_1(x) \sim \exp(-x^2/c)$, and we again will have integrability. We see that the boundedness condition (6.18) is a balance between the growth of the payoff function versus the properties of the density of the first factor.

6.2.2 The gamma

In the next proposition we calculate an expression for the Greek gamma.

Proposition 6.2.2. *Suppose the hypothesis of Prop. 6.2.1 holds, and in addition that there exists an integrable function v on \mathbb{R} such that*

$$|\mathbb{E}[f(x, X_2(T), \dots, X_n(T))]p_1''(x - \zeta(S(0)))| \leq v(x), \quad (6.20)$$

and that ζ is twice differentiable in $S(0)$. Then, the gamma is given by

$$\frac{\partial^2 C}{\partial S(0)^2} = e^{-rT} \mathbb{E} \left[h(S(T)) \left\{ (\zeta'(S(0)))^2 \frac{p_1''(X_1(T))}{p_1(X_1(T))} - \zeta''(S(0)) \partial \ln p_1(X_1(T)) \right\} \right]$$

Proof. From the proof in Prop. 6.2.1, we have that

$$\frac{\partial C}{\partial S(0)} = -\zeta'(S(0))e^{-rT} \int_{\mathbb{R}} \mathbb{E}[f(x, X_1(T), \dots, X_n(T))]p_1'(x - \zeta(S(0))) dx.$$

Appealing to the boundedness condition (6.20), we obtain the result by commuting the integration and differentiation using Thm. 2.27 in Folland [33]. \square

As we saw in the examples following Prop. 6.2.1, in most cases the function ζ is $\zeta(S(0)) = \ln S(0)$. For such a choice, $\zeta'(S(0)) = 1/S(0)$ and $\zeta''(S(0)) = -1/S^2(0)$, and thus the gamma becomes

$$\frac{\partial^2 C}{\partial S(0)^2} = \frac{1}{S^2(0)} e^{-rT} \mathbb{E} \left[h(S(T)) \left\{ \frac{p_1''(X_1(T))}{p_1(X_1(T))} + \partial \ln p_1(X_1(T)) \right\} \right].$$

Using the known density of $X_1(T)$ as in the examples for computation of the delta, we may give explicit expressions for the terms involving p_1 . In the case of the additive model by Benth, Kallsen, and Meyer-Brandis [9], we have $\zeta(S(0)) = \xi(T)S(0)$ for some known function $\xi(T)$. Then $\zeta''(S(0)) = 0$, thus we have an expression for the gamma given by

$$\frac{\partial^2 C}{\partial S(0)^2} = (\zeta'(S(0)))^2 e^{-rT} \mathbb{E} \left[h(S(T)) \frac{p_1''(X_1(T))}{p_1(X_1(T))} \right].$$

As remarked before in the case of the computation of the delta, we can argue here as well the possible use of the stationary distribution of X_1 exploiting the approximation that the stationary distribution represents to the original one of X_1 .

6.3 Forward prices, options on forwards, and their Greeks

Options are frequently written on forwards in the commodity markets. In fact, at NYMEX one trades in options on gas and oil futures, and at the Nordic electricity exchange Nord Pool European options are written on forwards and futures delivering electricity over specific periods. We therefore include a discussion on how our framework above may be incorporated to cover this situation as well.

Let us start our discussion with the two-factor model in (6.7), with $X(t)$ and $Y(t)$ being the base and spike components defined in (6.3) and (6.8), resp. To simplify the exposition, we ignore seasonality here. The forward price $F(t, \tau)$ of a contract at time $t \geq 0$, maturing at time $\tau \geq t$ is defined as

$$F(t, \tau) = \mathbb{E}[S(\tau) | \mathcal{F}_t], \quad (6.21)$$

see Benth, Saltyte Benth, and Koekebakker [11]. From Prop. 4.6 in the same reference, it follows that the forward price for the spot model in (6.7) becomes

$$F(t, \tau) = \Theta(\tau - t) \exp(X(t)e^{-\alpha(\tau-t)} + Y(t)e^{-\beta(\tau-t)}), \quad (6.22)$$

with

$$\ln \Theta(s) = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha s}) + \frac{\theta}{\alpha} (1 - e^{-\alpha s}) + \int_0^s \psi(-ie^{-\beta u}) du. \quad (6.23)$$

In order to establish the formula for the forward price, the jump process L must satisfy certain exponential integrability conditions, which can be found in Benth, Saltyte Benth, and Koekebakker [11]. We observe that the forward price is exactly represented into our machinery for calculating the delta of a spot.

The price of an option with exercise time $T \leq \tau$ and payoff function h is

$$C(F(0, \tau)) = e^{-rT} \mathbb{E}[h(F(T, \tau))].$$

We want to identify functions f and ζ along with factors X_1, \dots, X_n such that

$$h(F(T, \tau)) = f(X_1(t) + \zeta(F(0, \tau)), X_2(T), \dots, X_n(T)).$$

From the forward price in (6.22) and the explicit solutions of $X(T)$ and $Y(T)$, we find

$$\begin{aligned} F(T, \tau) &= \Theta(\tau - T) \exp \left(X(0)e^{-\alpha\tau} + \frac{\theta}{\alpha} e^{-\alpha\tau} (e^{\alpha T} - 1) + Y(0)e^{-\beta\tau} \right. \\ &\quad \left. + \int_0^T \sigma e^{-\alpha(\tau-s)} dW(s) + \int_0^T e^{-\beta(\tau-s)} dL(s) \right) \\ &= \frac{\Theta(\tau - T)}{\Theta(\tau)} \exp \left(\frac{\theta}{\alpha} e^{-\alpha\tau} (e^{\alpha T} - 1) \right) \\ &\quad \times \exp \left(\ln F(0, \tau) + \int_0^T \sigma e^{-\alpha(\tau-s)} dW(s) + \int_0^T e^{-\beta(\tau-s)} dL(s) \right), \end{aligned}$$

where we have used the fact that $F(0, \tau) = \Theta(\tau) \exp(X(0)e^{-\alpha\tau} + Y(0)e^{-\beta\tau})$. We then set $\zeta(x) = \ln x$,

$$\begin{aligned} X_1(T) &= \int_0^T \sigma e^{-\alpha(\tau-s)} dW(s), \\ X_2(T) &= \int_0^T e^{-\beta(\tau-s)} dL(s), \end{aligned}$$

and we get

$$f(x_1 + \zeta(F(0, \tau)), \dots, x_n) = h(\Psi(T, \tau) \exp(x_1 + \zeta(F(0, \tau)) + x_2)),$$

where

$$\Psi(T, \tau) = \frac{\Theta(\tau - T)}{\Theta(\tau)} \exp \left(\frac{\theta}{\alpha} e^{-\alpha\tau} (e^{\alpha T} - 1) \right).$$

Hence, we are in the framework of the previous subsection and we can apply Prop. 6.2.1 to obtain an expression for the delta of the option on F with payoff function h . We base the calculation on the density of $X_1(T)$, which is Gaussian with mean zero and variance $\sigma^2 e^{-2\alpha\tau} (e^{2\alpha T} - 1)/2\alpha$.

From Section 3.1 in Cartea and Figueroa [17], the forward price for the spot model in (6.9) is given by

$$\begin{aligned} F(t, \tau) &= \Theta(\tau - t) \exp \left(Y(0)e^{-\alpha\tau} - \frac{\lambda\sigma}{\alpha} e^{-\alpha\tau} (e^{\alpha t} - 1) + \int_0^t \sigma e^{-\alpha(\tau-s)} dW(s) \right. \\ &\quad \left. + \int_0^t e^{-\alpha(\tau-s)} \ln J dq(s) \right), \end{aligned} \tag{6.24}$$

where

$$\ln \Theta(s) = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha s}) - \frac{\lambda\sigma}{\alpha} (1 - e^{-\alpha s}) + \int_0^s \psi(\sigma_J, \alpha, s, u) l du - l s.$$

Here $\psi(\sigma_J, \alpha, s, u) = \exp\{-\frac{\sigma_J^2}{2} e^{-\alpha(s-u)} + \frac{\sigma_J^2}{2} e^{-2\alpha(s-u)}\}$. Therefore, we can write the function $F(T, \tau)$ as follows

$$F(T, \tau) = \frac{\Theta(\tau - T)}{\Theta(\tau)} \exp \left\{ -\frac{\lambda\sigma}{\alpha} e^{-\alpha\tau} (e^{\alpha T} - 1) \right\} \exp \left\{ \ln F(0, \tau) + \int_0^T \sigma e^{-\alpha(\tau-s)} dW(s) \right\}$$

$$+ \int_0^T e^{-\alpha(\tau-s)} \ln J dq(s) \Big\},$$

where $F(0, \tau) = \Theta(\tau) \exp(Y(0)e^{-\alpha\tau})$. We then set $\zeta(F(x)) = \ln x$,

$$X_1(T) = \int_0^T \sigma e^{-\alpha(T-s)} dW(s), \quad (6.25)$$

$$X_2(T) = \int_0^T \sigma e^{-\alpha(T-s)} \ln J dq(s),$$

and we get

$$f(x_1 + \zeta(F(0, \tau)), \dots, x_n) = h(\Psi(T, \tau) \exp(x_1 + \zeta(F(0, \tau)) + x_2)) ,$$

where

$$\Psi(T, \tau) = \frac{\Theta(\tau - T)}{\Theta(\tau)} \exp \left\{ -\frac{\lambda\sigma}{\alpha} e^{-\alpha\tau} (e^{\alpha T} - 1) \right\}.$$

As already indicated, forward contracts in power markets are not delivering the underlying commodity (that is, electricity) at a fixed time in the future, but rather over a given time period. This is due to the very nature of electricity as commodity. Hence, sometimes one refers to these contracts as *flow forwards*. The forward price $G(t, \tau_1, \tau_2)$ of a flow forward contract at time $t \geq 0$ with delivery in the period $[\tau_1, \tau_2]$, $\tau_1 \geq t$, is defined as

$$G(t, \tau_1, \tau_2) = \mathbb{E} \left[\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} S(u) du \mid \mathcal{F}_t \right], \quad (6.26)$$

see Benth, Saltyte Benth, and Koekebakker [11]. Note that the price is defined as the average spot price over delivery and not the aggregated spot. In reality, the aggregated spot is delivered, but, by market convention, the *forward price* is stated per time unit, that is, in MWh (Mega Watt hours) instead of MW. By commuting integration and expectation, we find

$$G(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} F(t, \tau) d\tau. \quad (6.27)$$

As it turns out, most models do not allow for analytical price formulas for flow forwards. For example, the exponential models discussed above yield in general only a price in terms of the integral in (6.27). However, considering the multi-factor spot model in (6.12) and in (6.13), one may derive explicit price dynamics for $G(t, \tau_1, \tau_2)$. Hereafter, we investigate this case and relate it to our analysis of the delta.

Consider the multi-factor model for the spot in (6.12) where the factors follow the dynamics in (6.13). To simplify the notation in our exposition, we suppose that the seasonality function is constant and equal to one, that is, $\Lambda(t) = 1$. Then, according to Prop. 4.14 in Benth, Saltyte Benth, and Koekebakker [11], the flow forward price is

$$G(t, \tau_1, \tau_2) = \Theta(t, \tau_1, \tau_2) + \sum_{i=1}^n Y_i(t) \tilde{\lambda}_i(t, \tau_1, \tau_2), \quad (6.28)$$

for a deterministic function Θ depending on the characteristics of the jump processes $L_i(t)$ (see Prop. 4.14 in Benth [11] for an explicit expression) and

$$\tilde{\lambda}_i(t, \tau_1, \tau_2) = \frac{1}{\lambda_i(\tau_2 - \tau_1)} \left(e^{-\lambda_i(\tau_1 - t)} - e^{-\lambda_i(\tau_2 - t)} \right). \quad (6.29)$$

We observe that

$$G(0, \tau_1, \tau_2) = \Theta(0, \tau_1, \tau_2) + \sum_{i=1}^n Y_i(0) \tilde{\lambda}_i(0, \tau_1, \tau_2).$$

Applying the explicit solution of the Ornstein-Uhlenbeck processes $Y_i(t)$, and reshuffling terms, we find

$$\begin{aligned} G(t, \tau_1, \tau_2) &= \tilde{\lambda}_1(t, \tau_1, \tau_2) e^{-\lambda_1 t} \left(\frac{G(0, \tau_1, \tau_2)}{\tilde{\lambda}_1(t, \tau_1, \tau_2)} + \int_0^t e^{\lambda_1 s} dL_1(s) - \frac{\Theta(0, \tau_1, \tau_2)}{\tilde{\lambda}_1(t, \tau_1, \tau_2)} \right) \\ &\quad + \sum_{i=2}^n Y_i(0) \left(\tilde{\lambda}_i(t, \tau_1, \tau_2) e^{-\lambda_i t} - \tilde{\lambda}_1(t, \tau_1, \tau_2) e^{-\lambda_1 t} \frac{\tilde{\lambda}_i(0, \tau_1, \tau_2)}{\tilde{\lambda}_1(0, \tau_1, \tau_2)} \right) \\ &\quad + \sum_{i=2}^n \tilde{\lambda}_i(t, \tau_1, \tau_2) e^{-\lambda_i t} \int_0^t e^{\lambda_i s} dL_i(s) + \Theta(t, \tau_1, \tau_2). \end{aligned}$$

Given a payoff function $h(G(T, \tau_1, \tau_2))$ for an option with exercise time $T \leq \tau_1$, we can read off the factors

$$X_i(T) = \int_0^T e^{\lambda_i s} dL_i(s),$$

for $i = 1, \dots, n$ and $\zeta(x) = x$. The function f is then easily defined, and we have obtained an expression coinciding with the kind we discuss above. The delta of the option $h(G(T, \tau_1, \tau_2))$ can then be calculated by appealing to Prop. 6.2.1 with appropriate change of notation. If the exact density function for the factor X_1 is not available, a reasonable way to proceed is given by its stationary distribution, as suggested for the computation of the delta and studied in Sect. 6.4.

6.4 Numerical examples

In this section we consider several numerical examples illustrating our conditional density approach. We use the popular finite difference approach for comparison. We look first at an example of a two-factor model where one of the factors has dynamics based on Brownian noise. This factor will have a normal density suitable for differentiation. In the second example we look at a model for the spot price which is stationary, and we apply the explicit knowledge of the stationary density to approximate the delta by conditioning. In the third example, we consider the two-factor model of Schwartz and Smith (6.4) and study the effect of different choices of factors to be used for delta computation. Finally, we consider the Cartea and Figueroa model (6.9) and compute the delta of a call option on the forward written in this model.

6.4.1 Example 1

In our first numerical example, we consider the two-factor model $S(t)$ given by

$$S(t) = S(0) \exp(X(t) + Y(t)),$$

where $X(t)$ is a mean-reverting process given by equation (6.3). This form of a spot price model is rather typical in commodity markets (see for example Meyer-Brandis and Tankov [51]). For numerical illustration, we choose the parameters of $X(t)$ to be $\theta = 0$, $\alpha = 0.099$, and $\sigma = 0.032$. We hence ignore any market price of risk (modelled through θ), and suppose that the half-life of X is $\ln(2)/\alpha \approx 7$, that is, it takes around seven days for any deviation in X to be halved, on average. This is a rather typical half-life for the base component in a spot price of electricity. The volatility corresponds to approx 50% annually. The process $Y(t)$ is supposed to account for the spikes, and is given by equation (6.8). We choose $\beta = 0.23$ and we set L as a compound Poisson with jump frequency $\lambda = 20/250$ and exponentially distributed jump size, with mean 0.2. The half-life is hence 3 days, and spikes occur on average 20 times a year. The mean spike size is an increase of 20% in the price. This model has parameters which are reasonable in view of EEX spot prices (see later examples).

We consider a function h being the payoff of a call option with strike $K = 100$:

$$h(S(T)) = \max(S(T) - 100, 0).$$

In Figure 6.1, we show the resulting delta for an at-the-money option $S(0) = 100$ and exercise time $T = 20$ days. To estimate the expectation operator, we have used Monte Carlo simulation. Along the horizontal axis, we have the number of simulations (in 10^4) used in the estimation of the expectation operator. The solid line shows the derivative using the finite difference method, that is,

$$\frac{\partial C}{\partial S(0)} \approx \frac{C(S(0) + \delta) - C(S(0))}{\delta},$$

where $\delta = 0.01$. The broken line shows the delta using the conditional density method. Common random numbers are used in the Monte Carlo simulation. We clearly see that the conditional density method has higher variance than the finite difference approach, and thus a slower convergence.

In Figure 6.2, we consider a digital option with payoff function

$$h(S(T)) = 1_{(100, \infty)}(S(T)),$$

where $T = 20$ days. The solid line shows the delta using a finite difference method with $\delta = 0.01$ and the broken line shows the delta using the conditional density method. We observe that in this case the conditional density method has much lower variance, and therefore converges faster than the finite difference method. The rather high variation yielding uncertain Monte Carlo estimates that results from the finite difference method, is well-known for payoff functions which are not differentiable. The conditional density method has in this case a much more stable performance. We would get the same conclusions

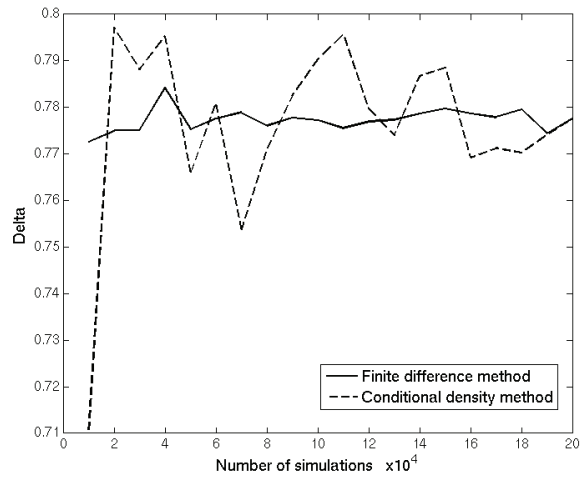


Figure 6.1: Simulation of the delta for a call option

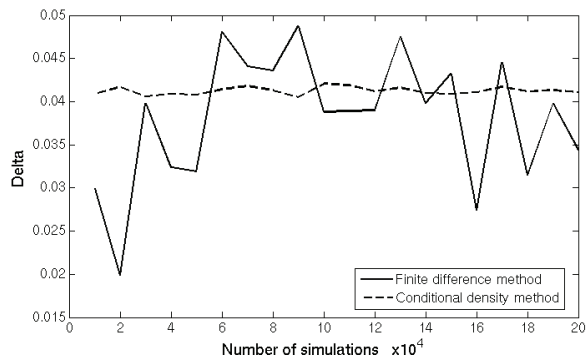


Figure 6.2: Simulation of the delta for a digital option

looking at options on forwards. Again the conditional density method would converge faster for singular payoffs. Moreover, this result will carry over to the gamma, the second derivative of the option with respect to the underlying spot price. The computation of the gamma essentially involves the second derivative of the payoff function, and thus the case of the gamma of a call option would show similar features as the case of the delta of a digital option. In this case the conditional density method would outperform the finite difference method.

6.4.2 Example 2

The second example that we consider is a special case of the additive model of Benth, Kallsen, and Meyer-Brandis [9]. Let the spot price be given as a two-factor model,

$$S(t) = X(t) + Y(t), \quad S(0) > 0.$$

Here, the process $Y(t)$ is given by

$$Y(t) = -\lambda_2 Y(t)dt + dL_2(t), \quad Y(0) = 0,$$

where L_2 is a compound Poisson process with intensity μ and exponentially distributed jumps with parameter ν . The process $X(t)$ is a so called $\Gamma(a, b)$ -OU process. Namely, it is a Lévy process following the dynamics

$$dX(t) = -\lambda_1 X(t)dt + dL_1(t), \quad X(0) = S(0),$$

where $L_1(t)$ is a subordinator, admitting a stationary distribution which is here $\Gamma(a, b)$ (see Thm 17.5 in Sato [59] and Thm 1 in Barndorff-Nielsen and Shephard [5]).

The problem now is to compute the delta of an option written on the spot. We have not given any explicit density here, so apparently the conditional density method is not working. However, we know that X (in fact also Y) has a stationary distribution, and we can apply this for the conditional density method in order to derive the delta, at least approximately.

To check out the validity of such an approximation, we need to be able to simulate from the processes in the spot model. To simulate a $\Gamma(a, b)$ -OU process, we first remark that $L_1(t)$ is actually a compound Poisson process with intensity parameter a and exponential jump distribution with parameter b (see Example 2 in Section 2 in Barndorff-Nielsen and Shephard [5]). Then from

$$X(t) = e^{-\lambda_1 t} X(0) + \int_0^t e^{\lambda_1(s-t)} dL_1(s), \quad (6.30)$$

we see that in order to simulate $X(t)$, we need to simulate a Poisson process with intensity $\lambda_1 a$ at the discrete times $t_n = n\Delta t$, $n = 0, 1, \dots$. Then, we set

$$x(n\Delta t) = e^{-\lambda_1 \Delta t} x((n-1)\Delta t) + \sum_{N((n-1)\Delta t)+1}^{N(n\Delta t)} z_n e^{-u_n \lambda_1 \Delta t},$$

where z_n are independent $\text{Exp}(b)$ random numbers and u_n are independent uniform random numbers.

Consider the payoff of a call option $h(S(T)) = \max(S(T) - K, 0)$. We apply the conditional density method in the following way. First of all, we observe that the stochastic integral

$$\int_0^t e^{\lambda_1(s-t)} dL_1(s) = X(t) - e^{-\lambda_1 t} X(0)$$

has an asymptotic distribution being $\Gamma(a, b)$ when t goes to infinity, since $e^{-\lambda_1 t} X(0)$ goes to 0 when t goes to infinity. Denoting by Z a random variable which is $\Gamma(a, b)$ -distributed, we consider

$$\tilde{S}(t) = e^{-\lambda_1 t} X(0) + Z + Y(t),$$

which is asymptotically equal in distribution to $S(t)$.

In the notation of Prop 6.2.1, we have the factors, $X_1(T) = Z$, $X_2(T) = Y(T)$, $\zeta(S(0)) = e^{-\lambda_1 T} S(0)$ and $h(S(T)) = f(X_1(T) + \zeta(S(0)), X_2(T))$. Therefore, for any density $p_1(x)$ defined on the positive half axis, we have in particular that

$$\begin{aligned} \frac{\partial C}{\partial S(0)} &\approx \frac{\partial}{\partial S(0)} e^{-rT} \mathbb{E}[f(X_1(T) + \zeta(S(0)), X_2(T))] \\ &= \frac{\partial}{\partial S(0)} e^{-rT} \int_{\zeta(S(0))}^{+\infty} \mathbb{E}[f(x, X_2(T))] p_1(x - \zeta(S(0))) dx \\ &= e^{-rT} \int_{\zeta(S(0))}^{+\infty} \mathbb{E}[f(x, X_2(T))] \frac{\partial p_1}{\partial S(0)}(x - \zeta(S(0))) dx - \mathbb{E}[f(\zeta(S(0)), X_2(T))] p_1(0), \end{aligned}$$

where in the latter equality, we used the fact that

$$\frac{\partial}{\partial y} \int_y^{+\infty} g(x, y) dx = \int_y^{+\infty} \frac{\partial g}{\partial y}(x, y) dx - g(y, y).$$

Therefore

$$\begin{aligned} \frac{\partial C}{\partial S(0)} &\approx e^{-rT} \mathbb{E}[f(X(T) + \zeta(S(0)), Y(T))(-\zeta'(S(0))) \frac{\partial}{\partial x} \log p_1(X(T))] \\ &\quad - \mathbb{E}[f(\zeta(S(0)), Y(T))] p_1(0). \end{aligned}$$

In our study, in the case of a $\Gamma(a, b)$, the density is given by $p_1(x) = x^{a-1} \frac{e^{-x/b}}{\Gamma(a)b^a}$, $x > 0$, $a, b > 0$. Note that when $0 < a < 1$, p_1 is not defined in 0, while when $a = 1$, it is equal to $\frac{1}{\Gamma(1)b}$ and finally for $a > 0$, it is equal to 0. The expression for the delta is then given by

$$\frac{\partial C}{\partial S(0)} \approx e^{-rT} \mathbb{E}\left[h(e^{-\lambda_1 T} X(0) + Z + Y(T)) e^{-\lambda_1 T} \left(b - \frac{a-1}{Z}\right)\right].$$

This will be our approximation of the delta based on the conditional density method.

To make a numerical example which is relevant for energy markets, we note that Benth, Kiesel, and Nazarova [10] showed empirically that the spot model fits the Phelix Base electricity price index at the European Power Exchange (EEX) very well. In their paper, they estimated the parameters in the suggested model to be $a = 13.3009$, $b = 8.5689$,

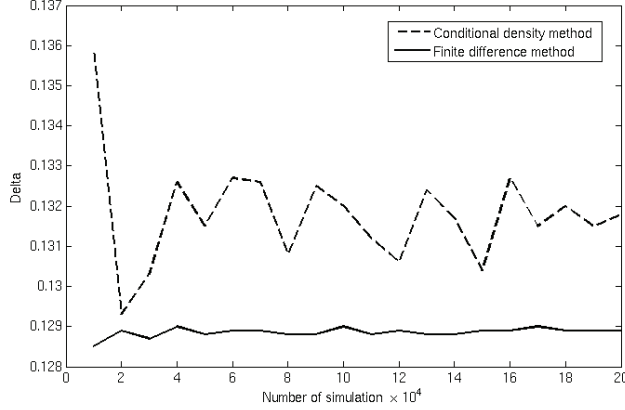


Figure 6.3: Simulation of the delta for a call option

$\lambda_1 = 0.2008$, $\lambda_2 = 0.3333$, $\mu = 20/250$, and $\nu = 0.2$. We use these estimates in our example, however, we let the seasonality function be constant equal to one for simplicity. We remark that the inclusion of a seasonal function is straightforward. In our numerical examples we ignore any risk premium.

In Figure 6.3, we show the resulting derivative for $S(0) = 5$, strike $K = 1.5$ (for an at the money), exercise time $T = 10$ days, and interest rate $r = 0$. To estimate the expectation operators in the conditional density and finite difference methods, we use a Monte Carlo simulation technique with common random numbers. Along the horizontal axis, we have the number of simulations (in 10^4) used in the estimation of the expectation operators. The solid line shows the derivative using a finite difference method, that is,

$$\frac{\partial C}{\partial S(0)} \approx \frac{C(S(0) + \delta) - C(S(0))}{\delta}, \quad (6.31)$$

with $\delta = 0.01$. In the expression (6.31), we used the fact that

$$\begin{aligned} C(S(0) + \delta) &= e^{-rT} \mathbb{E} \left[\max \left(e^{-\lambda_1 T} (X(0) + \delta) + \int_0^T e^{\lambda_1(s-t)} dL_1(s) + Y(T) - K, 0 \right) \right] \\ &= e^{-rT} \mathbb{E} \left[\max(e^{-\lambda_1 T} \delta + X(T) + Y(T) - K, 0) \right]. \end{aligned}$$

The broken line shows the delta using the conditional density method. Again we find that the finite difference method converges faster for the delta of a call option, not unexpectedly. But, interestingly, the approximation based on conditional density seems to be reasonably good. Based on 600,000 samples in a Monte Carlo simulation, the "true value" resulting from the conditional density method is 0.132 with three decimals of accuracy. The finite difference method converges slightly below 0.129, giving an upward bias of approximately 2% for the conditional density approximation relative to the finite difference method.

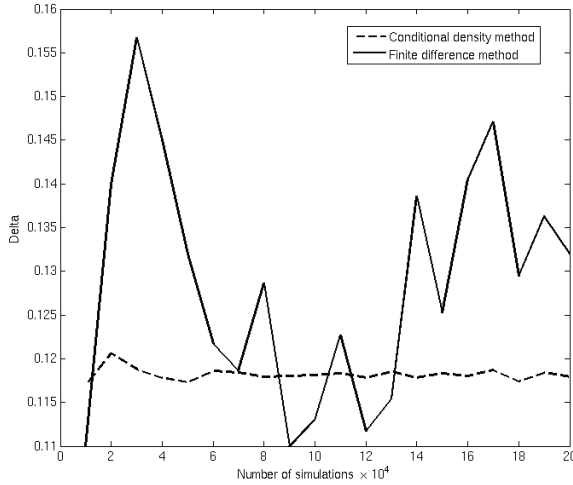


Figure 6.4: Simulation of the delta for a digital option

Motivated by the above, we go further to study a digital option and its delta. Consider a digital option with payoff $h(S(T) = 1_{(K, \infty)}(S(T)))$, where the strike is $K = 2$, $S(0) = 5$, exercise time $T = 10$ days and interest rate $r = 0$. In Figure 6.4, the solid line shows the delta using a finite difference method with $\delta = 0.01$ and the broken line shows the approximation using the conditional density method. We observe that as in Example 1, the conditional density method converges faster for singular payoffs. Based on 600,000 outcomes, the finite difference method gave the result 0.129 with three decimals of accuracy. The conditional density method is now downward biased, and the error of the conditional density method relative to the finite difference is approximately 7%.

How well the approximation based on the conditional density method works is depending on how far from stationarity the $X(t)$ factor is. We have looked at options with only 10 days left to exercise, and one may argue this is a rather short time for the model to be in stationarity. This taken into account, one may say that the approximation is rather good despite the deviation of around 7% relative to the finite difference method. How fast the model goes into stationary is also depending on the speed of mean reversion and the size and frequency of jumps. In conclusion, the approximating method may provide an attractive alternative to other methods like the finite difference method since it converges so much faster.

6.4.3 Example 3

In this example, we deal with the two-factor model of Schwartz and Smith given in (6.4). In Subsection 6.2.1, we compute the delta of the option written on the spot price S .

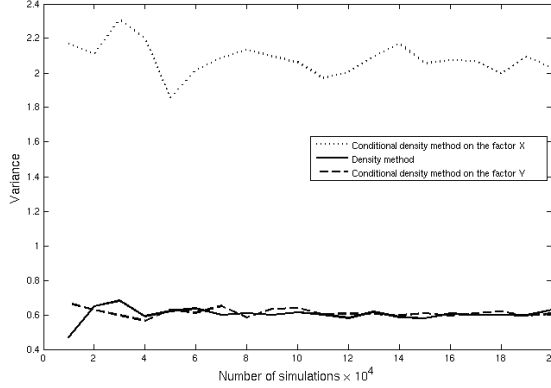


Figure 6.5: Simulation of the variance of the payoff times the weight

We obtain formulas for the delta of the form $\mathbb{E}[h(S(T)\pi)]$, where h is the payoff of a call option with strike K and π is a random variable. Here π is depending on X when we use the conditional density method on the random variable X , depending on Y when we use the conditional density method on Y , or depending on $X + Y$ when we use the density method. The different expressions for the random variable π are given by

$$\begin{aligned}\pi &:= \pi_X = \frac{2\alpha}{S(0)\sigma^2(1 - e^{-2\alpha T})} \left(X(T) - X(0)e^{-\alpha T} - \frac{\theta}{\alpha}(1 - e^{-\alpha T}) \right), \\ \pi &:= \pi_Y = \frac{1}{S(0)\eta^2 T} (Y(T) - Y(0) - \mu T), \\ \pi &:= \pi_{X+Y} = \frac{2\alpha}{S(0)(\sigma^2(1 - e^{-2\alpha T}) + 2\alpha\eta^2 T)} \left(X(T) + Y(T) \right. \\ &\quad \left. - e^{-\alpha T} X(0) - \frac{\theta}{\alpha}(1 - e^{-\alpha T}) - Y(0) - \mu T \right).\end{aligned}$$

In our numerical example, we use parameter estimates based on the Enron data taken from Schwartz and Smith [61]. The state variable and parameters for the first factor X are estimated to be $X(0) = 0.119$, $\alpha = 1.19$, $\sigma = 0.158$, and $\theta = -0.014$. The state variable and parameters of the second factor Y are given by $Y(0) = 2.857$, $\mu = -0.0386$, $\eta = 0.115$. In Figure 6.5, we show the resulting variance $\text{Var}[h(S(T)\pi)]$, for the different π 's with $T = 10$ days and $K = S(0) = 17$. We have used a Monte Carlo simulation to estimate the variance operator. Along the horizontal axis we have the number of simulations. The dotted line shows $\text{Var}[h(S(T)\pi_X)]$, the broken line $\text{Var}[h(S(T)\pi_Y)]$, and the solid line $\text{Var}[h(S(T)\pi_{X+Y})]$.

Clearly, the conditional density method has a very high variance when we apply the weight π_X compared to the weights π_Y and π_{X+Y} . The two latter are approximately equal in performance. This shows that the choice of factors is critical for the speed of

convergence. Indeed, the stationary part X has less influence on the spot in the long-mean while both Y obviously $X + Y$ describe better the spot for future time T . The shape of the weights π_Y and π_{X+Y} are stronger dependent on S than π_X and thereby have the effect of reducing the variance of $h(S(T))\pi$.

6.4.4 Example 4

In our final example, we consider the forward price (6.24) derived from the spot model of Cartea and Figueroa in (6.9). We compute the delta of call option written on the forward, applying the conditional density method on the process X_1 given by (6.25). Therefore, invoking Prop. 6.2.1 and conditioning on $X_1(T)$, we find the following expression for the delta

$$\frac{\partial C}{\partial F(0, \tau)} = \frac{e^{-rT} 2\alpha}{\sigma^2 e^{-2\alpha\tau} (e^{2\alpha T} - 1) F(0, \tau)} \mathbb{E}[h(F(T, \tau) X_1(T))].$$

We use a parameter estimate from the European Energy Exchange (EEX) taken from Benth, Kiesel and Nazarova [10]. The parameters of the process Y given by (6.10) are as follows $Y(0) = 50$, $\alpha = 0.2255$, $\sigma = 0.039025$, $\sigma_J = 0.010996$, $l = 5.67$, $\mu_J = -0.5 \times 10^{-4}$ and $\lambda = -0.002481$.

In Figure 6.6, we show the resulting delta for exercise time $T = 10$ days, strike $K = 50$, and $\tau = 20$. To estimate the expectation operator, we have used Monte Carlo simulation. Along the horizontal axis, we have the number of simulations (in 10^4) used in the estimation of the expectation operator. The solid line shows the derivative using the finite difference method that is

$$\frac{\partial C}{\partial F(0, \tau)} \approx \frac{C(F(0, \tau) + \delta) - C(F(0, \tau))}{\delta},$$

where $\delta = 0.01$. The broken line shows the delta using the conditional density method.

Again we observe that the conditional density method has bigger variance than the finite difference method. However, noteworthy is the upward bias in the finite difference computed delta. The finite difference method is a numerical differentiation and as such is an approximation of the true value. Hence, it will always give a biased estimate of the delta, where the bias will depend on the discretization δ . In our previous examples this bias has not been so pronounced. However, the combination of a non-smooth payoff function with jumps in the underlying dynamics seems to produce a larger bias in some circumstances. Note that we have

$$\frac{C(F(0, \tau) + \delta) - C(F(0, \tau))}{\delta} = \mathbb{E} \left[\frac{h(F(0, \tau) Z(T, \tau) + \delta Z(T, \tau)) - h(F(0, \tau) Z(T, \tau))}{\delta} \right],$$

where $Z(T, \tau)$ is defined via the expression for the forward in (6.24). Suppose now that h is a smooth function, and apply the mean-value Theorem from calculus to get

$$\frac{C(F(0, \tau) + \delta) - C(F(0, \tau))}{\delta} = \mathbb{E} [h'(U) Z(T, \tau)],$$

where U is a random variable such that $F(0, \tau) Z(T, \tau) \leq U \leq (F(0, \tau) + \delta) Z(T, \tau)$. Having jumps in the model usually gives distributions of the spot which are more spread

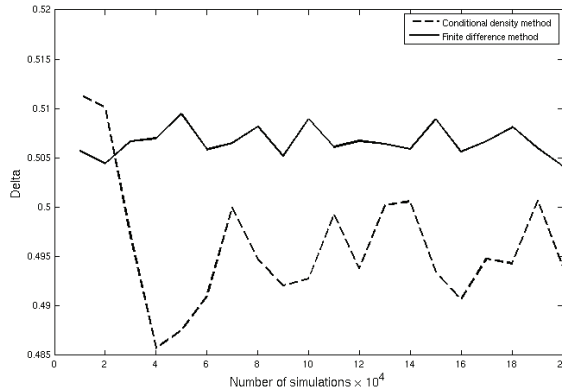


Figure 6.6: Simulation of the delta of a call option on a forward

out than Gaussian models, and thus leading to a bigger range for the random variable U . The bias may become enlarged when h is not differentiable (for a call option it is not differentiable at the strike price). Our experience is that combining this with a jump process, the bias becomes significant if one is not carefully choosing δ sufficiently small. A smaller δ , on the other hand, requires more simulations.

We remark that also in Example 1 we had jumps in the spot model, in a very similar fashion. Note however, that the jump part of the model in Example 1 had a very fast mean-reversion, so at exercise of the option one has a very little effect left of this factor. In the model of Cartea and Figueroa, the jumps revert at the same relatively slow speed as the “normal variations” given by the Brownian part. Hence, the impact of jumps becomes more significant.

6.5 Conclusions

We have analysed theoretically and numerically the conditional density method for computing Greeks of options in the context of energy markets. This method is particularly suitable in energy since most of the price models can be represented by a multi-factor dynamics, where at least one of the factors has a known density. This is exploited to derive expectation functionals which gives unbiased estimates of the Greek in question without having to differentiate the payoff function.

The conditional density method works for options written on many of the popular energy spot models, including the Schwartz and Smith model (see [61]), the Cartea and Figueroa model (see [17]), and the factor model in Benth, Kallsen, and Meyer-Brandis [9] to mention some. Also, options on forwards are easily included in our approach. We have a focus on the Greeks delta and gamma, although the methodology is easily extended to other Greeks.

Several numerical examples are provided, where we benchmark the conditional density method against the popular finite difference approach. The latter approach is based on a numerical approximation of the derivative, and gives biased estimates of the Greek in question. The Greeks resulting from both methods are numerically computed based on Monte Carlo simulations (the latter with common random numbers). In general, the Greek computed using the finite difference approach converges faster than the conditional density method for call and put options, whereas for digitals the conclusion is reverse. One can explain this by the singularity of the payoff function, since for digital options the payoff is non-differentiable, while calls and puts have smoother payoffs in the sense of only being non-differentiable at the strike price. Based on this, we conclude that the conditional density method provides an attractive alternative for options with singular payoffs, but also for computation of the gamma of a call or put option.

For some models, like the Schwartz-Smith spot price dynamics, one can use several factors in the conditional density method, as well as the standard density method. We provide a numerical study of the variance for the different choices, and show that selection of factors crucially influence the variation in the random variable to be computed. In fact, the conditional density method is equally efficient as the density method when choosing the non-stationary factor to condition on.

Our numerical studies suggest that the finite difference method may have a significant bias in the case of jumps in the underlying model. As most of the models for spot and forwards in energy markets naturally include jumps (to model spikes, say in gas and electricity prices), this is an important issue. The conditional density method gives an unbiased estimate, and does not face this problem.

Finally, we exploit the stationarity in energy prices to propose approximations of the Greeks for models where one may not have explicitly known densities in one or more factors. This is relevant for models where we know the stationary distribution (see the factor model in Benth, Kallsen, and Meyer-Brandis [9]).

For other models which do not fit into the framework of multi-factor dynamics as considered here, one may adapt the Malliavin approach proposed in Chapter 5. This is left for future studies, where one may consider regime-switching models or the state dependent jump-diffusion model by Geman and Roncoroni [40].

Computation of the delta in multidimensional jump-diffusion setting with applications to stochastic volatility models

This chapter is extracted from the paper "Computation of the delta in multidimensional jump-diffusion setting with applications to stochastic volatility models" by Asma Khedher, available at *E-print*, April (2011), Department of Mathematics, University of Oslo, Norway, submitted for publication.

In this chapter, we study the robustness of option prices to model variation in a multidimensional jump-diffusion framework. In particular we consider price dynamics in which small variations are modeled either by a Poisson random measure with infinite activity or by a Brownian motion. We consider both European and Exotic options and we study their deltas using two approaches: the Malliavin method and the Fourier method. We prove robustness of the deltas to model variation. We apply these results to the study of stochastic volatility models for the underlying and the corresponding options.

The chapter is organized as follows. In section 7.1 we make a short introduction about multidimensional Lévy processes. In section 7.2 we study the computation of the delta and the related analysis of robustness to the model. Section 7.3 deals with the computation of the delta in stochastic volatility models and the robustness of the BN-S model.

7.1 Some mathematical preliminaries

In this chapter we consider a d -dimensional Lévy process $L = (L^{(1)}(t), \dots, L^{(d)}(t))^*$, $0 \leq t \leq T$. Here $*$ denotes the transpose of a given vector or a given matrix and $L^{(i)}$, $1 \leq i \leq d$, are d Lévy processes. Let $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$. We consider d independent Poisson random measures

$$N(dt, dz) = (N^{(1)}(dt, dz), \dots, N^{(d)}(dt, dz)), \quad z \in \mathbb{R}, \quad (7.1)$$

an \mathbb{R}^m -Brownian motion $W = (W^{(1)}(t), \dots, W^{(m)}(t))^*$, $0 \leq t \leq T$, a vector $A \in \mathbb{R}^d$, and a symmetric non-negative definite matrix $\Sigma \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$. From the Lévy-Itô decomposition

(equation (2.1)), we have

$$L(t) = At + \Sigma W(t) + Z(t) + \lim_{\varepsilon \downarrow 0} \tilde{Z}_\varepsilon(t), \quad (7.2)$$

where $Z(t) = (Z^{(1)}(t), \dots, Z^{(d)}(t))^*$, $\tilde{Z}_\varepsilon(t) = (\tilde{Z}^{(1)}_\varepsilon(t), \dots, \tilde{Z}^{(d)}_\varepsilon(t))^*$ such that

$$Z^{(i)}(t) = \int_0^t \int_{|z|>1} z N^{(i)}(ds, dz), \quad \forall \quad 1 \leq i \leq d,$$

$$\tilde{Z}^{(i)}_\varepsilon(t) = \int_0^t \int_{\varepsilon < |z| \leq 1} z \tilde{N}^{(i)}(ds, dz), \quad \forall \quad 1 \leq i \leq d$$

and the Lévy measure of L is given by $\ell(dz) = (\ell_1(dz), \dots, \ell_d(dz))$. Notice that the Lévy processes $L^{(i)}$, $1 \leq i \leq d$, are independent. The dependent case will be studied in a future work.

We introduce the following notation for the variation of the Lévy process L close to the origin $\sigma^2(\varepsilon) = (\sigma_1^2(\varepsilon), \dots, \sigma_d^2(\varepsilon))^*$, where

$$\sigma_i^2(\varepsilon) := \int_{|z|<\varepsilon} z^2 \ell_i(dz), \quad 0 < \varepsilon \leq 1, \quad 1 \leq i \leq d. \quad (7.3)$$

Since every Lévy measure $\ell_i(dz)$ integrates z^2 in an open interval around zero, we have that $\sigma_i^2(\varepsilon)$, $1 \leq i \leq d$, are finite for any $\varepsilon > 0$. Note that the $\sigma_i^2(\varepsilon)$ is the variance of the jumps of $L^{(i)}$ smaller than ε in the case $L^{(i)}$ is symmetric and has mean zero. By dominated convergence $\sigma_i^2(\varepsilon)$, $1 \leq i \leq d$, converge to zero when $\varepsilon \downarrow 0$.

Recall the Lévy-Itô decomposition of a Lévy process L and introduce now an approximating Lévy process (in law)

$$L_\varepsilon(t) := At + \Sigma W(t) + \sigma(\varepsilon)B(t) + Z(t) + \tilde{Z}_\varepsilon(t), \quad (7.4)$$

where $\sigma^2(\varepsilon)$ is as in (7.3) and B is a one-dimensional Brownian motion independent of L (which in particular means independent of W). From the definition of $\tilde{Z}_\varepsilon^{(i)}$, $1 \leq i \leq d$, we see that we have substituted the small jumps (compensated by their expectation) in $L^{(i)}$ by a Brownian motion scaled with $\sigma_i(\varepsilon)$, the standard deviation of the compensated small jumps. We have the following result taken from Chapter 5.

Proposition 7.1.1. *Let the process L , respectively L_ε , be defined as in equation (7.2), respectively (7.4). Then, for every t ,*

$$\lim_{\varepsilon \rightarrow 0} L_\varepsilon^{(i)}(t) = L^{(i)}(t) \quad \mathbb{P} - a.s., \quad \forall \quad 1 \leq i \leq d.$$

In fact, the limit above also holds in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E} [|L_\varepsilon^{(i)}(t) - L^{(i)}(t)|] \leq 2\sigma_i(\varepsilon)\sqrt{t}, \quad \forall \quad 1 \leq i \leq d.$$

We shall make use of the approximation and its convergence properties in our analysis.

7.2 Robustness of option prices and their deltas

7.2.1 Robustness of option prices

In this section we consider the robustness of jump-diffusions given by the solution of stochastic differential equations of the form $X(t) = (X^{(1)}(t), \dots, X^{(d)}(t))$, where

$$\begin{aligned} X^{(i)}(t) &= x_i + \int_0^t \alpha_i(X(s-)) ds + \int_0^t \sum_{j=1}^m \beta_{ij}(X(s-)) dW^{(j)}(s) \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \gamma_i(X(s-), z) \tilde{N}^{(i)}(ds, dz), \quad 1 \leq i \leq d. \end{aligned} \quad (7.5)$$

Here $x_i \in \mathbb{R}$, α_i, β_{ij} are measurable functions $\mathbb{R}^d \rightarrow \mathbb{R}$, and γ_i is a measurable function $\mathbb{R}^d \times \mathbb{R}_0 \rightarrow \mathbb{R}$. We assume, moreover, that the coefficient functions $\alpha_i(x)$ and $\beta_{ij}(x)$ have linear growth and are Lipschitz continuous. Each $\gamma_i(x, z)$ is of the form $\gamma_i(x, z) = \delta_i(x)g_i(z)$, where the (stochastic) factor $\delta_i(x)$ has linear growth and is Lipschitz continuous and the (deterministic) factors $g_i(z)$ satisfy

$$G^2(\infty) = \sum_{i=1}^d \int_{\mathbb{R}_0} g_i^2(z) \ell_i(dz) < \infty,$$

which will ensure that $\forall 1 \leq i \leq d$, $X^{(i)}(t)$ has finite variance. We also define

$$G_i^2(\varepsilon) = \int_{|z| < \varepsilon} g_i^2(z) \ell_i(dz), \quad 1 \leq i \leq d$$

and

$$G^2(\varepsilon) = \sum_{i=1}^d \int_{|z| < \varepsilon} g_i^2(z) \ell_i(dz),$$

for later use.

Introduce the approximating jump-diffusion dynamics where the small jumps part in (7.5) has been substituted by the Brownian motion B independent of W and appropriately scaled, namely $X_\varepsilon(t) = (X_\varepsilon^{(1)}(t), \dots, X_\varepsilon^{(d)}(t))$, where

$$\begin{aligned} X_\varepsilon^{(i)}(t) &= x_i + \int_0^t \alpha_i(X_\varepsilon(s-)) ds + \int_0^t \sum_{j=1}^m \beta_{ij}(X_\varepsilon(s-)) dW^{(j)}(s) \\ &\quad + \int_0^t \left(\int_{|z| < \varepsilon} (\gamma_i^2(X_\varepsilon(s-), z) \ell_i(dz))^{1/2} dB(s) + \int_0^t \int_{|z| \geq \varepsilon} \gamma_i(X_\varepsilon(s-), z) \tilde{N}^{(i)}(ds, dz) \right) \\ &= x_i + \int_0^t \alpha_i(X_\varepsilon(s-)) ds + \int_0^t \sum_{j=1}^m \beta_{ij}(X_\varepsilon(s-)) dW^{(j)}(s) \\ &\quad + \int_0^t G_i(\varepsilon) \delta_i(X_\varepsilon(s-)) dB(s) + \int_0^t \int_{|z| \geq \varepsilon} \gamma_i(X_\varepsilon(s-), z) \tilde{N}^{(i)}(ds, dz). \end{aligned} \quad (7.6)$$

The existence and uniqueness of the solutions $X(t)$ and $X_\varepsilon(t)$ are ensured by the following theorem collected from Ikeda and Watanabe [46] (Thm 9.1. Chap IV):

Theorem 7.2.1. *Let U be an open set in \mathbb{R}_0 , α be a measurable function $\mathbb{R}^d \rightarrow \mathbb{R}^d$, β be a measurable function $\mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^m$, and γ be a measurable function $\mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ such that, for some positive constant K ,*

$$\|\alpha(x)\|^2 + \|\beta(x)\|^2 + \int_U \|\gamma(x, z)\|^2 \ell(dz) \leq K(1 + \|x\|^2), \quad x \in \mathbb{R}^d, \quad (7.7)$$

$$\|\alpha(x) - \alpha(y)\|^2 + \|\beta(x) - \beta(y)\|^2 + \int_U \|\gamma(x, z) - \gamma(y, z)\|^2 \ell(dz) \leq K\|x - y\|^2, \quad x, y \in \mathbb{R}^d. \quad (7.8)$$

Then there exists a unique d -dimensional \mathcal{F}_t -adapted right-continuous process $X(t)$ with left-hand limits which satisfies the following stochastic differential equation

$$X^{(i)}(t) = x_i + \int_0^t \alpha_i(X(s-)) ds + \int_0^t \sum_{j=1}^m \beta_{ij}(X(s-)) dW^{(j)}(s) \quad (7.9)$$

$$+ \int_0^t \int_U \gamma_i(X(s-), z) \tilde{N}^{(i)}(ds, dz), \quad 1 \leq i \leq d. \quad (7.10)$$

In Prop. 5.2.1, we prove the convergence of $X_\varepsilon(t)$ to $X(t)$, where $X(t)$ is a one-dimensional stochastic differential equation. In the same way, we prove the following result

Proposition 7.2.1. *For every $0 \leq t \leq T < \infty$, we have*

$$\sum_{i=1}^d \|X^{(i)}(t) - X_\varepsilon^{(i)}(t)\|_2^2 \leq CG^2(\varepsilon),$$

where $X^{(i)}$ and $X_\varepsilon^{(i)}$, $\forall 1 \leq i \leq d$, are solutions of (7.5) and (7.6), respectively and C , is a positive constant depending on T , but independent of ε .

From Proposition 7.2.1, we can deduce the following result.

Proposition 7.2.2. *Let $X^{(i)}$ and $X_\varepsilon^{(i)}$, $\forall 1 \leq i \leq d$, be solutions of (7.5) and (7.6), respectively. For every $0 \leq t \leq T < \infty$, we have*

$$\sum_{i=1}^d \left\| \int_0^T \{X^{(i)}(t) - X_\varepsilon^{(i)}(t)\} dt \right\|_2^2 \leq C'G^2(\varepsilon),$$

where C' is a positive constant depending on T , but independent of ε .

Proof. By Hölder inequality and Proposition 7.2.1, $\forall 1 \leq i \leq d$, we have

$$\begin{aligned} \sum_{i=1}^d \left\| \int_0^T \{X^{(i)}(t) - X_\varepsilon^{(i)}(t)\} dt \right\|_2^2 &\leq \sum_{i=1}^d T \mathbb{E} \left[\int_0^T \{X^{(i)}(t) - X_\varepsilon^{(i)}(t)\}^2 dt \right] \\ &\leq \sum_{i=1}^d T \int_0^T \mathbb{E}[\{X^{(i)}(t) - X_\varepsilon^{(i)}(t)\}^2] dt \\ &\leq T^2 CG^2(\varepsilon) \end{aligned}$$

and the result follows. \square

Moreover, we have the following robustness of option prices.

Corollary 7.2.1. *Suppose $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Lipschitz continuous function and X and X_ε solve (7.5) and (7.6), resp. Then, for every $0 \leq t \leq T < \infty$, there exists two positive constants C and C' depending on T but independent of ε such that*

$$|\mathbb{E}[f(X_\varepsilon(t))] - \mathbb{E}[f(X(t))]| \leq CG(\varepsilon)$$

and

$$|\mathbb{E}\left[f\left(\int_0^T X_\varepsilon(t)dt\right)\right] - \mathbb{E}\left[f\left(\int_0^T X(t)dt\right)\right]| \leq C'G(\varepsilon).$$

Proof. Letting K be the Lipschitz constant of f , we have from the Jensen inequality,

$$\begin{aligned} |\mathbb{E}[f(X_\varepsilon(t))] - \mathbb{E}[f(X(t))]| &\leq K\mathbb{E}[|X_\varepsilon(t) - X(t)|] \\ &\leq K\left(\sum_{i=1}^d \mathbb{E}[|X_\varepsilon^{(i)}(t) - X^{(i)}(t)|^2]\right)^{\frac{1}{2}}. \end{aligned}$$

The latter follows from the Cauchy-Schwarz inequality. Applying Prop. 7.2.1, the result follows. Moreover, we have

$$|\mathbb{E}\left[f\left(\int_0^T X_\varepsilon(t)dt\right)\right] - \mathbb{E}\left[f\left(\int_0^T X(t)dt\right)\right]| \leq K\mathbb{E}\left[\left\|\int_0^T \{X_\varepsilon(t) - X(t)\}dt\right\|\right]$$

Hence, from the Cauchy-Schwarz inequality and Prop. 7.2.2, the result follows. \square

7.2.2 Computation of the Delta and robustness

In this section we present the Malliavin approach to compute the delta for option prices based on a multidimensional jump-diffusion market model. We consider the approach studied in Chapter 5 which is based on a separability assumption. We assume that $m = d$ and that the diffusion matrix $\beta \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ has an inverse β^{-1} and satisfies the uniform ellipticity condition

$$\exists \eta > 0; \quad \xi^* \beta^*(x) \beta(x) \xi \geq \eta |\xi|^2, \quad \text{for any } \xi, x \in \mathbb{R}^d. \quad (7.11)$$

Separability approach. Let $\mathcal{F}_t^{\tilde{N}} = \sigma\{\int_0^s \int_U (\tilde{N}^{(1)}(du, dz), \dots, \tilde{N}^{(d)}(du, dz))\}; \quad s \leq t, \quad U \in \mathcal{B}(\mathbb{R}_0)\}$. Assume that $\forall 1 \leq i \leq d$, α_i , β_i , and γ_i are continuously differentiable functions with bounded derivatives and consider Markov jump diffusions, $X^{(i)}$ of the form (7.5), for which we have a continuously differentiable functions $h_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ with bounded derivative in the first argument such that

$$X^{(i)}(t) = h_i(X^{c(i)}(t), X^{J(i)}(t)), \quad X^{(i)}(0) = x_i, \quad 1 \leq i \leq d. \quad (7.12)$$

Here $X^{c(i)}$ satisfies a stochastic differential equation

$$dX^{c(i)}(t) = \alpha_{ci}(X^{c(i)}(t))dt + \sum_{j=1}^d \beta_{cij}(X^{c(i)}(t))dW^{(j)}(t),$$

$$X^{c(i)}(0) = x_i = h_i(X^{c(i)}(0), X^{J(i)}(0)), \quad (7.13)$$

with continuously differentiable coefficients α_{ci} , β_{cij} , while $X^{J(i)}$ is adapted to the natural filtration $\mathcal{F}^{\tilde{N}}$ of the compensated compound Poisson process \tilde{N} . In particular, $X^{J(i)}$ does not depend on x_i . The jump-diffusion process of type (7.12) is called *separable*.

We associate with the process X^c , a process V given by

$$V(t) = I + \int_0^t \alpha'_c(X^c(s))V(s)ds + \int_0^t \sum_{i=1}^d \beta'_{ci}(X^c(s))V(s)dW^{(i)}(s), \quad (7.14)$$

where I is the identity matrix, $\alpha_c = (\alpha_{c(1)}, \dots, \alpha_{c(d)})^*$, β_{ci} is the i -th column vector of β_c , and prime denotes derivatives. The process V is called *the first variation process* for X^c and we have

$$V(t) = \nabla X^c(t).$$

We provide an example of a jump-diffusion dynamics satisfying our assumptions. Consider a jump-diffusion of the form

$$\begin{aligned} dX^{(1)}(t) &= \alpha_1(X^{(2)}(t), \dots, X^{(d)}(t))X^{(1)}(t^-)dt + \beta_1(X^{(2)}(t), \dots, X^{(d)}(t))X^{(1)}(t^-)dW(t) \\ &\quad + \int_{\mathbb{R}_0} (e^z - 1)X^{(1)}(t^-)\tilde{N}(dt, dz), \quad X^{(1)}(0) = x_1, \\ dX^{(i)}(t) &= \alpha_i(X^{(2)}(t), \dots, X^{(d)}(t))X^{(i)}(t^-)dt + \beta_i(X^{(2)}(t), \dots, X^{(d)}(t))X^{(i)}(t^-)dW(t), \\ &\quad X^{(i)}(0) = x_i, \quad i = 2, \dots, d, \end{aligned}$$

where α_i and β_i are constants. We introduce the process $X^{c(1)}(t)$ defined by

$$\begin{aligned} dX^{c(1)}(t) &= \left(\alpha_1(X^{c(2)}(t), \dots, X^{c(d)}(t)) + \int_{\mathbb{R}_0} (1 + z - e^z)\ell(dz) \right) X^{c(1)}(t)dt \\ &\quad + \beta_1(X^{c(2)}(t), \dots, X^{c(d)}(t))X^{c(1)}(t)dW(t), \quad X^{c(1)}(0) = x_1, \\ dX^{c(i)}(t) &= \alpha_i(X^{c(2)}(t), \dots, X^{c(d)}(t))X^{c(i)}(t)dt + \beta_i(X^{c(2)}(t), \dots, X^{c(d)}(t))X^{c(i)}(t)dW(t), \\ &\quad X^{c(i)}(0) = x_i, \quad i = 2, \dots, d, \end{aligned}$$

Then by applying the Itô formula to

$$\begin{aligned} \hat{X}^{(1)}(t) &= e^{\tilde{Z}(t)}X^{c(1)}(t), \quad \tilde{Z}(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz), \\ \hat{X}^{(i)}(t) &= X^{c(i)}(t), \end{aligned}$$

we can prove that $\hat{X}(t) = X(t)$ a.e.

We define the payoff function $f = f(X(t_1), \dots, X(t_n))$ to be a square integrable function discounted from maturity T and evaluated at the times t_1, \dots, t_n . We are interested in differentiating expectations of the form

$$v(x) = \mathbb{E}[f(X(t_1), \dots, X(t_n))]$$

with respect to the state of the underlying asset. The following result is the extension of the Theorem 5.3.1 for the computation of the delta for a European option written in a multidimensional jump-diffusion. Davis and Johansson [24] derived expressions for the delta written in multidimensional jump-diffusion processes in which the jump part is modeled by independent Poisson processes. In our case we consider more general jump-diffusions. However, the proof follows the same steps than the proof of Proposition 3.4 in Davis and Johansson [24] since we use the Malliavin derivative only in the continuous part in the jump-diffusion dynamics.

Theorem 7.2.2. *Let X be a diffusion of the form (7.5). We assume the uniform ellipticity condition (7.11) and the separability condition. Define*

$$\Gamma = \left\{ a \in L^2[0, T] \mid \int_0^{t_i} a(t) dt = 1, \quad \forall i = 1, \dots, n \right\}.$$

Then for $a \in \Gamma$ and $f(X(t_1), \dots, X(t_n))$ square integrable, we have

$$\Delta = (\nabla v(x))^* = \mathbb{E} \left[f(X(t_1), \dots, X(t_n)) \int_0^T a(t) (\beta_c^{-1}(X^c(t)) V(t))^* dW(t) \right],$$

where V is given by (7.14).

Now we consider the case of an Asian option with payoff of the form $f\left(\int_0^T X(t) dt\right)$. In the following theorem we give the formula for the derivative with respect to the initial condition in dimension one.

Theorem 7.2.3. *Let X be a diffusion of the form (7.5) with $d = 1$. Let $f(\omega) = f(Z(\omega))$, where $Z(T) = \int_0^T X(t) dt$. We assume the uniform ellipticity condition (7.11) and the separability condition. Then for $f(Z(\omega)) \in L^2(\Omega)$,*

$$\Delta = \mathbb{E} \left[f \left(\int_0^T X(t) dt \right) \delta \left(2V^2(t) \frac{\partial X(t)}{\partial X^c(t)} \left\{ \beta_c(X^c(t)) \int_0^T \frac{\partial X(u)}{\partial X^c(u)} V(u) du \right\}^{-1} \right) \right],$$

where V is given by (7.14).

Proof. Assume that $f \in C_K^\infty(\mathbb{R})$. Then

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E} \left[f \left(\int_0^T X(t) dt \right) \right] &= \mathbb{E} \left[f' \left(\int_0^T X(t) dt \right) \int_0^T \frac{\partial X(t)}{\partial x} dt \right] \\ &= \mathbb{E} \left[f' \left(\int_0^T X(t) dt \right) \int_0^T \frac{\partial X(t)}{\partial X^c(t)} V(t) dt \right], \end{aligned} \quad (7.15)$$

where V is the first variation process for X^c . Consider a random variable $\eta \in L^2(\Omega \times [0, T])$. Then by the chain rule (Proposition 5.1.1), we have

$$\mathbb{E} \left[\int_0^T D_{u,0} f \left(\int_0^T X(t) dt \right) \eta(u) du \right]$$

$$\begin{aligned}
&= \mathbb{E} \left[\int_0^T \left\{ f' \left(\int_0^T X(t) dt \right) D_{u,0} \left(\int_0^T X(t) dt \right) \eta(u) \right\} du \right] \\
&= \mathbb{E} \left[f' \left(\int_0^T X(t) dt \right) \int_0^T \left(\int_0^T D_{u,0} X(t) dt \right) \eta(u) du \right] \\
&= \mathbb{E} \left[f' \left(\int_0^T X(t) dt \right) \int_0^T \eta(u) \left(\int_u^T \frac{\partial X(t)}{\partial X^c(t)} V(t) V^{-1}(u) \beta_c(X^c(u)) dt \right) du \right] \\
&= \mathbb{E} \left[f' \left(\int_0^T X(t) dt \right) \int_0^T \left(\int_0^t \eta(u) V^{-1}(u) \beta_c(X^c(u)) du \right) \frac{\partial X(t)}{\partial X^c(t)} V(t) dt \right]
\end{aligned}$$

We choose

$$\eta(u) = 2V^2(u) \frac{\partial X(u)}{\partial X^c(u)} \beta_c^{-1}(X^c(u)) \left(\int_0^T \frac{\partial X(t)}{\partial X^c(t)} V(t) dt \right)^{-1}.$$

Using the fact that $2 \int_0^T \int_0^v f(u) f(v) du dv = (\int_0^T f(s) ds)^2$, we get

$$\begin{aligned}
&\mathbb{E} \left[f' \left(\int_0^T X(t) dt \right) \int_0^T \left(\int_0^t \eta(u) V^{-1}(u) \beta_c(X^c(u)) du \right) \frac{\partial X(t)}{\partial X^c(t)} V(t) dt \right] \\
&= \mathbb{E} \left[f' \left(\int_0^T X(t) dt \right) \int_0^T \frac{\partial X(t)}{\partial X^c(t)} V(t) dt \right].
\end{aligned}$$

The result, for $f \in C_K^\infty(\mathbb{R})$, follows from the duality formula (Proposition 5.1.2). We can extend this formula to $f(Z(w)) \in L^2(\Omega)$ following the Proposition 2.2.2. \square

We next address the question of robustness of the delta with respect to approximations of the small jumps by an appropriately scaled continuous martingale. As in Chapter 5, it turns out that this question can be efficiently answered by means of *Fourier transform*.

Assume that $f \in L^1(\mathbb{R}^d)$, the space of integrable functions on \mathbb{R}^d . The Fourier transform of f is defined by

$$\widehat{f}(u) = \int_{\mathbb{R}^d} f(y) e^{iu \cdot y} dy, \quad (7.16)$$

where u and y are two d -dimensional vectors and $u \cdot y$ is the standard scalar product in \mathbb{R}^d . Suppose in addition that $\widehat{f} \in L^1(\mathbb{R}^d)$. Then the inverse Fourier transform is well-defined, and we have

$$f(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu \cdot y} \widehat{f}(u) du. \quad (7.17)$$

Following Carr and Madan [21], we calculate,

$$\begin{aligned}
\mathbb{E}[f(X_\varepsilon^x(t))] &= \int_{\mathbb{R}^d} \left\{ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iy \cdot u} \widehat{f}(u) du \right\} P_{X_\varepsilon^x(t)}(dy) \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} e^{-iy \cdot u} P_{X_\varepsilon^x(t)}(dy) \right\} \widehat{f}(u) du \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(u) \mathbb{E} [e^{-iu \cdot X_\varepsilon^x(t)}] du, \quad (7.18)
\end{aligned}$$

where $P_{X_\varepsilon(t)}(dy)$ is the distribution of $X_\varepsilon(t) = X_\varepsilon^x(t)$, the solution of (5.6) with $X_\varepsilon(0) = X_\varepsilon^x(0) = x$. Fubini-Tonelli's Theorem (see Folland [33]) is applied to commute the integrations. Similarly, we get for $X(t) = X^x(t)$ being the solution of (7.5) with $X(0) = X^x(0) = x$,

$$\mathbb{E}[f(X^x(t))] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(u) \mathbb{E}[e^{-iu \cdot X^x(t)}] du. \quad (7.19)$$

Thus, in order to study the delta, we need to be able to move differentiation inside the inverse Fourier transform. But, furthermore, we must have accessible the derivative of $X_\varepsilon^x(t)$ and $X^x(t)$ with respect to x . Before moving on with the robustness of deltas, we study this.

Introduce $Y(t) = (Y^{(i,j)}(t))_{i=1,\dots,d,j=1,\dots,d} = (\frac{\partial X^{(i)}(t)}{\partial x_j})_{i=1,\dots,d,j=1,\dots,d}$, where each $Y^{(i,j)}$ satisfies the following stochastic differential equation

$$Y^{(i,j)}(t) = \varrho + \int_0^t \sum_{k=1}^d \partial_k \alpha_i(X^x(s-)) Y^{(k,j)}(s-) ds \quad (7.20)$$

$$\begin{aligned} &+ \int_0^t \sum_{k=1}^d \sum_{n=1}^d \partial_k \beta_{in}(X^x(s-)) Y^{(k,j)}(s-) dW^{(n)}(s) \\ &+ \int_0^t \int_{\mathbb{R}_0^d} \sum_{k=1}^d \partial_k \gamma_i(X^x(s-), z) Y^{(k,j)}(s-) \widetilde{N}^{(i)}(ds, dz), \end{aligned} \quad (7.21)$$

where $\varrho = 1$ if $i = j$ and $\varrho = 0$ if $i \neq j$. Since the derivatives of α_i, β_{ik} and γ_i are assumed to be bounded, it follows from Thm. 7.2.1 that there exists a unique solution $Y(t)$ of (7.20). From Thm 40 in Chapter V of Protter [56], it follows that $X^x(t)$ is differentiable with respect to x , and that

$$\nabla X^x(t) = Y(t). \quad (7.22)$$

By the same considerations, $X_\varepsilon^x(t)$ is differentiable with respect to x , and

$$\nabla X_\varepsilon^x(t) = Y_\varepsilon(t), \quad (7.23)$$

with $Y_\varepsilon(t) = (Y_\varepsilon^{(i,j)}(t))_{i=1,\dots,d,j=1,\dots,d} = (\frac{\partial X_\varepsilon^{(i)}(t)}{\partial x_j})_{i=1,\dots,d,j=1,\dots,d}$, where each $Y_\varepsilon^{(i,j)}$ satisfies the following stochastic differential equation

$$Y_\varepsilon^{(i,j)}(t) = \varrho + \int_0^t \sum_{k=1}^d \partial_k \alpha_i(X_\varepsilon^x(s-)) Y_\varepsilon^{(k,j)}(s-) ds \quad (7.24)$$

$$\begin{aligned} &+ \int_0^t \sum_{k=1}^d \sum_{n=1}^d \partial_k \beta_{in}(X_\varepsilon^x(s-)) Y_\varepsilon^{(k,j)}(s-) dW^{(n)}(s) \\ &+ \int_0^t G_i(\varepsilon) \sum_{k=1}^d \partial_k \delta_i(X_\varepsilon^x(s-)) Y_\varepsilon^{(k,j)}(s-) dB(s) \end{aligned} \quad (7.25)$$

$$+ \int_0^t \int_{|z| \geq \varepsilon} \sum_{k=1}^d \partial_k \gamma_i(X_\varepsilon^x(s-), z) Y_\varepsilon^{(k,j)}(s-) \widetilde{N}^{(i)}(ds, dz). \quad (7.26)$$

In the next Proposition we derive the expressions for the delta based on X and X_ε using the Fourier method.

Proposition 7.2.3. *Let $X^x(t)$ and $Y(t)$ be solutions of (7.5) and (7.20), resp., and $X_\varepsilon^x(t)$ and $Y_\varepsilon(t)$ of (7.6) and (7.24), resp. Let $u\hat{f}(u) \in L^1(\mathbb{R}^d)$. Then, for $0 \leq t \leq T$,*

$$\begin{aligned}\nabla \mathbb{E}[f(X^x(t))] &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(u) \mathbb{E}[-iuY(t)e^{-iu \cdot X^x(t)}] du \\ \nabla \mathbb{E}[f(X_\varepsilon^x(t))] &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(u) \mathbb{E}[-iuY_\varepsilon(t)e^{-iu \cdot X_\varepsilon^x(t)}] du \\ \nabla \mathbb{E}\left[f\left(\int_0^T X^x(t)dt\right)\right] &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(u) \mathbb{E}\left[\left\{\int_0^T -iuY(t)dt\right\}e^{-iu \cdot \int_0^T X^x(t)dt}\right] du \\ \nabla \mathbb{E}\left[f\left(\int_0^T X_\varepsilon^x(t)dt\right)\right] &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(u) \mathbb{E}\left[\left\{\int_0^T -iuY_\varepsilon(t)dt\right\}e^{-iu \cdot \int_0^T X_\varepsilon^x(t)dt}\right] du\end{aligned}$$

Proof. First, by dominated convergence, we can move the gradient inside the integral and inside the expectation operator on the right-hand side in (7.19). Next, differentiating, we obtain straightforwardly the results since $Y(t) = \nabla X^x(t)$. We follow the same argument for $X_\varepsilon^x(t)$, $\int_0^T X^x(t)dt$, and $\int_0^T X_\varepsilon^x(t)dt$. \square

Finally, we state our result on robustness. The proof follows the same steps of the proof of Proposition 5.3.3.

Proposition 7.2.4. *Let $u\hat{f}(u) \in L^1(\mathbb{R}^d)$. For $0 \leq t \leq T$, it holds that*

$$\lim_{\varepsilon \downarrow 0} \nabla \mathbb{E}[f(X_\varepsilon^x(t))] = \nabla \mathbb{E}[f(X^x(t))]$$

and

$$\lim_{\varepsilon \downarrow 0} \nabla \mathbb{E}\left[f\left(\int_0^T X_\varepsilon^x(t)dt\right)\right] = \nabla \mathbb{E}\left[f\left(\int_0^T X^x(t)dt\right)\right]$$

7.3 Application to stochastic volatility models

Stochastic volatility models describe the joint evolution of the underlying asset price and its variance. Let us first consider the following general stochastic volatility model.

$$\begin{cases} dX(t) &= \mu X(t-)dt + u(Y(t-))X(t-)dW^{(1)}(t-) + \int_{\mathbb{R}_0} (e^z - 1)X(t-)\tilde{N}(dt, dz), \\ dY(t) &= b(t-, Y(t-))dt + v(t-, Y(t-))dW^{(2)}(t-) + \int_{\mathbb{R}_0} \beta(z)\tilde{N}(dt, dz), \end{cases} \quad (7.27)$$

Here $X(0) = x$, $Y(0) > 0$, $\mu \in \mathbb{R}$, b and v are Lipschitz continuous and differentiable functions on $[0, T] \times \mathbb{R}$, u is a nonnegative function Lipschitz continuous and differentiable on \mathbb{R} , β is a function on \mathbb{R} , \tilde{N} is a compound Poisson process, and $W^{(1)}$ and $W^{(2)}$ are two correlated standard Brownian motions. We have

$$dW^{(1)}(t)dW^{(2)}(t) = \rho dt, \quad \rho \in (-1, 1).$$

Therefore there exists a Brownian motion \widetilde{W} , independent of $W^{(1)}$ and $W^{(2)}$ such that we can express $W^{(1)}$ in terms of \widetilde{W} and $W^{(2)}$ as follows

$$W^{(1)}(t) = \rho W^{(2)}(t) + \sqrt{1 - \rho^2} \widetilde{W}(t).$$

The process X plays the role of the stock price process, while $u(Y)$ is the volatility process. Introduce the following stochastic differential equation

$$\begin{aligned} dX^{c,Y}(t) &= \left(\mu + \int_{\mathbb{R}_0} (1 + z - e^z) \ell(dz) \right) X^{c,Y}(t) dt + u(Y(t)) X^{c,Y}(t) \rho dW^{(2)}(t) \\ &\quad + u(Y(t)) X^{c,Y}(t) \sqrt{1 - \rho^2} d\widetilde{W}(t), \\ X^{c,Y}(0) &= x. \end{aligned}$$

We denote by $V^Y = \frac{\partial X^{c,Y}(t)}{\partial x}$. Then we have the following proposition.

Proposition 7.3.1. *Consider the general stochastic volatility model (7.27). Then for $a \in \Gamma$ and $f \in L^2(\Omega)$, we have*

$$\begin{aligned} \Delta &= E \left[f(X(T), Y(T)) \left(\int_0^T \frac{a(t) V^Y(t)}{u(Y(t)) X^{c,Y}(t) (1 - \rho^2)} dW^{(1)}(t) \right. \right. \\ &\quad \left. \left. - \int_0^T \frac{\rho a(t) V^Y(t)}{u(Y(t)) X^{c,Y}(t) (1 - \rho^2)} dW^{(2)}(t) \right) \right]. \end{aligned}$$

Proof. We denote by $D^{\widetilde{W}}$, the Malliavin derivative with respect to the Brownian motion \widetilde{W} . Thus, by Thm.2.2.1 in Nualart [54], we have

$$\begin{aligned} D_t^{\widetilde{W}} X^{c,Y}(T) &= u(Y(t)) X^{c,Y}(t) \sqrt{1 - \rho^2} + \int_t^T \left(\mu + \int_{\mathbb{R}_0} (1 + z - e^z) \ell(dz) \right) D_t^{\widetilde{W}} X^{c,Y}(s) ds \\ &\quad + \int_t^T D_t^{\widetilde{W}} \left(u(Y(s)) X^{c,Y}(s) \rho \right) dW^{(2)}(s) \\ &\quad + \int_t^T D_t^{\widetilde{W}} \left(u(Y(s)) X^{c,Y}(s) \sqrt{1 - \rho^2} \right) d\widetilde{W}(s). \end{aligned} \tag{7.28}$$

As the process Y depends only on the Brownian motion $W^{(2)}$ and a jump part, then we have

$$\begin{aligned} D_t^{\widetilde{W}} X^{c,Y}(T) &= u(Y(t)) X^{c,Y}(t) \sqrt{1 - \rho^2} + \int_t^T \left(\mu + \int_{\mathbb{R}_0} (1 + z - e^z) \ell(dz) \right) D_t^{\widetilde{W}} X^{c,Y}(s) ds \\ &\quad + \int_t^T u(Y(s)) \rho D_t^{\widetilde{W}} X^{c,Y}(s) dW^{(2)}(s) \\ &\quad + \int_t^T u(Y(s)) \sqrt{1 - \rho^2} D_t^{\widetilde{W}} X^{c,Y}(s) d\widetilde{W}(s). \end{aligned}$$

Therefore $D_t^{\widetilde{W}} X^{c,Y}(T) = V^Y(T) (V^Y(t))^{-1} (u(Y(t)) X^{c,Y}(t) \sqrt{1 - \rho^2})$. However, the delta is given by

$$\Delta = \frac{\partial}{\partial x} \mathbb{E} \left[f(X(T), Y(T)) \right] = \mathbb{E} \left[f'(X(T), Y(T)) \frac{\partial X(T)}{\partial x} \right]$$

$$= \mathbb{E} \left[f'(X(T), Y(T)) \frac{\partial X(T)}{\partial X^{c,Y}(T)} \frac{\partial X^{c,Y}}{\partial x} \right].$$

The process X can be written as $X(t) = e^{\tilde{Z}(t)} X^{c,Y}(t)$, where $\tilde{Z}(t) = \int_{\mathbb{R}_0} z \tilde{N}(dt, dz)$. Therefore, by the chain rule (Proposition 5.1.1), we have

$$D_{t,0}X(T) = \frac{\partial X(T)}{\partial X^{c,Y}} D_t^{\tilde{W}} X^{c,Y}(T).$$

We replace $D_t^{\tilde{W}} X^{c,Y}(T)$ by its expression, we get

$$D_{t,0}X(T) = \frac{\partial X(T)}{\partial X^{c,Y}} V^Y(T) (V^Y(t))^{-1} (u(Y(t)) X^{c,Y}(t) \sqrt{1 - \rho^2}).$$

Hence

$$\frac{\partial X(T)}{\partial X^{c,Y}(T)} V^Y(T) = D_{t,0}X(T) V^Y(t) (u(Y(t)) X^{c,Y}(t) \sqrt{1 - \rho^2})^{-1}.$$

Therefore, we get the expression for the delta as follows

$$\Delta = E \left[f(X(T), Y(T)) \int_0^T a(t) V^Y(t) (u(Y(t)) X^{c,Y}(t) \sqrt{1 - \rho^2})^{-1} d\tilde{W}(t) \right], \quad (7.29)$$

where $a(t) \in \Gamma$.

As for the robustness, we can approximate the stochastic volatility model (7.27) by the following

$$\begin{cases} dX_\varepsilon(t) = \mu X_\varepsilon(t-) dt + u(Y_\varepsilon(t-)) X_\varepsilon(t-) dW^{(1)}(t-) \\ \quad + \left(\int_{|z| < \varepsilon} (e^z - 1)^2 \ell(dz) \right)^{\frac{1}{2}} X_\varepsilon(t-) dB^{(1)}(t) + \int_{|z| \geq \varepsilon} (e^z - 1) X_\varepsilon(t-) \tilde{N}(dt, dz), \\ dY_\varepsilon(t) = b(t-, Y_\varepsilon(t-)) dt + v(t-, Y_\varepsilon(t-)) dW^{(2)}(t) \\ \quad + \left(\int_{|z| < \varepsilon} \beta^2(z) \ell(dz) \right)^{\frac{1}{2}} dB^{(2)}(t) + \int_{|z| \geq \varepsilon} \beta(z) \tilde{N}(dt, dz), \end{cases} \quad (7.30)$$

where $(B^{(1)}, B^{(2)})$ is a Brownian motion independent of $(W^{(1)}, W^{(2)})$. By Proposition 7.2.1, we have the convergence when ε goes to 0 of the equation (7.30) to the equation (7.27) in $L^2(\Omega)$. The convergence of the option price and its delta when ε goes to 0 follows from Corollary 7.2.1 and Proposition 7.2.4. \square

As an example, we give a slight generalization of the Heston model (see Heston [44]). That is we consider a Heston model with jumps in the underlying asset price.

Heston model. The Heston model is given by

$$\begin{aligned} dX(t) &= rX(t-)dt + \sqrt{Y(t)}X(t-)dW^{(1)}(t) \\ &\quad + \int_{\mathbb{R}_0} (e^z - 1)X(t-) \tilde{N}(dt, dz), \quad X(0) = x_1, \\ dY(t) &= k(\theta - Y(t))dt + \eta \sqrt{Y(t)}dW^{(2)}(t), \quad Y(0) > 0. \end{aligned}$$

r is a deterministic risk free interest rate, θ is a long-term variance, k is a mean-reverting rate, and η is referred to the volatility of the variance. We assume that $2k\theta \geq \eta$. The volatility in this model is the square root of the mean reverting process Y , introduced by Cox, Ingersoll, and Ross [16]. The square root function is neither differentiable in zero nor globally Lipschitz. In a paper by Alos and Ewald [2], the uniqueness and existence of solution is proved. Moreover, it is proved that $\sqrt{Y(t)}$ is Malliavin differentiable (Corollary 4.2 in Alos and Ewald [2]). We consider the process $X^{c,Y}$ given by

$$\begin{aligned} dX^{c,Y} &= (r + \int_{\mathbb{R}_0} (1 + z - e^z)\ell(dz))X^{c,Y}(t)dt + \sqrt{Y(t)}\rho X^{c,Y}(t)dW^{(2)}(t) \\ &+ \sqrt{Y(t)}X^{c,Y}(t)\sqrt{1-\rho^2}d\widetilde{W}(t). \end{aligned}$$

This process is Malliavin differentiable with respect to the Brownian motion \widetilde{W} therefore Proposition 7.3.1 still applies and taking $u(Y(t)) = \sqrt{Y(t)}$, $V^Y(t) = X^{c,Y}(t)/x_1$, and $a(t) = 1/T$, the delta is given by

$$\Delta = E\left[f(X(T), Y(T))\frac{1}{x_1 T} \left(\int_0^T \frac{dW^{(1)}(t)}{\sqrt{Y(t)}(1-\rho^2)} - \frac{\rho}{1-\rho^2} \int_0^T \frac{dW^{(2)}(t)}{\sqrt{Y(t)}} \right)\right].$$

Notice that the weights which we found involve the volatility \sqrt{Y} and the Brownian motions $W^{(1)}$ and $W^{(2)}$. This is similar to the weights found in Davis and Johansson [24].

A second example is the Heston model with jumps in the volatility (see Matytsin [52] and Sepp [59]).

Heston model with jumps in the volatility. We consider the following stochastic differential equation

$$\begin{aligned} dX(t) &= rX(t-)dt + \sqrt{Y(t)}X(t-)dW^{(1)}(t) \\ &+ (\alpha - 1)X(t-)(dN(t) - \lambda dt), \quad X(0) = x_1, \\ dY(t) &= k(\theta - Y(t-))dt + \eta\sqrt{Y(t-)}dW^{(2)}(t) + \beta dJ(t), \quad Y(0) = x_2, \end{aligned}$$

where N is a Poisson process with constant intensity λ and J is a Poisson process independent of N . β is a constant. We assume that $2k\theta \geq \eta$. We consider $\widehat{X}(t) = \alpha^{N(t)}X^{c,Y}(t)$, where

$$dX^{c,Y}(t) = (\lambda(1-\alpha)+r)X^{c,Y}(t)dt + \sqrt{Y(t)}X^{c,Y}(t)\rho dW^{(2)}(t) + \sqrt{Y(t)}X^{c,Y}(t)\sqrt{1-\rho^2}d\widetilde{W}(t).$$

Applying the Itô formula to \widehat{X} , we have $\widehat{X} = X$, a.s. By Corollary 4.2 in Alos and Ewald [2] and Theorem 2.2 in Nualart [54], the process $X^{c,Y}$ is Malliavin differentiable with respect to the Brownian motion \widetilde{W} . Therefore applying Proposition 7.3.1, with $u(Y(t)) = \sqrt{Y(t)}$, $V^Y(t) = X^{c,Y}(t)/x_1$, and $a(t) = 1/T$, the delta is given by

$$\Delta = E\left[f(X(T), Y(T))\frac{1}{x_1 T} \left(\int_0^T \frac{dW^{(1)}(t)}{\sqrt{Y(t)}(1-\rho^2)} - \frac{\rho}{1-\rho^2} \int_0^T \frac{dW^{(2)}(t)}{\sqrt{Y(t)}} \right)\right].$$

Stability of option prices (the BNS model)

We consider the following BNS model,

$$\begin{cases} dX(t) &= (\mu + \beta Y(t))dt + \sqrt{Y}(t)dW(t) + \rho dZ(t), & X(0) = x, \\ dY(t) &= -\lambda Y(t)dt + dZ(t), & Y(0) > 0, \end{cases} \quad (7.31)$$

where the parameters μ, β, ρ , and λ are real constants with $\lambda > 0$ and $\rho \leq 0$. $Z = Z(t)$, $0 \leq t \leq T$ is a subordinator (i.e., increasing Lévy process). We assume that Z has no deterministic drift and its Lévy measure has density $\omega(z)$, so that the cumulant transform $k(\theta) = \log \mathbb{E}[e^{\theta Z_1}]$, where it exists takes the form

$$k(\theta) = \int_{\mathbb{R}_+} (e^{\theta z} - 1)\omega(z)dz.$$

We denote by N the random measure associated with the jumps of Z . We consider a parameter λ_ε , $0 < \varepsilon < 1$, such that

$$\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = \lambda.$$

Notice that in this case by triangular inequality we have

$$|\lambda_\varepsilon| \leq |\lambda| + |\lambda_\varepsilon - \lambda|.$$

In particular when ε is sufficiently small, we have $|\lambda_\varepsilon - \lambda| \leq 1$. Therefore $|\lambda_\varepsilon| \leq a$, where $a = 1 + |\lambda|$. Thus, we have the following approximation for the BNS model

$$\begin{cases} dX_\varepsilon(t) &= (\mu + \beta Y_\varepsilon(t))dt + \sqrt{Y_\varepsilon(t)}dW(t) + \rho dZ(t), & X_\varepsilon(0) = x, \\ dY_\varepsilon(t) &= -\lambda_\varepsilon Y_\varepsilon(t)dt + dZ(t), & Y_\varepsilon(0) > 0. \end{cases} \quad (7.32)$$

In the following, we study the robustness of the BN-S model and the associated option price. The computation of the delta is studied in Benth, Groth, and Wallin [12].

Lemma 7.3.1. *The system given by (7.32) converges to (7.31) almost surely when ε goes to 0.*

Proof. The process Y_ε is given by

$$Y_\varepsilon(t) = e^{-\lambda_\varepsilon t} Z(0) + \int_0^t e^{\lambda_\varepsilon(s-t)} dZ(s). \quad (7.33)$$

As $e^{\lambda_\varepsilon s} \leq e^{aT}$, then by dominated convergence theorem, we can take the limit inside the integral in (7.33) and we have the almost sure convergence of the process Y_ε to the process Y when ε goes to 0. The process X_ε is given by

$$X_\varepsilon(t) = x + \int_0^t (\mu + \beta Y_\varepsilon(s))ds + \int_0^t \sqrt{Y_\varepsilon(s)}dW(s) + \rho Z(t). \quad (7.34)$$

As we have $|Y_\varepsilon| \leq Z(0) + e^{aT} Z(T)$, then by dominated convergence theorem, we can take the limit inside the integral in (7.34) and the result follows. \square

We consider a European option written on $S(t) = e^{X(t)}$, $0 \leq t \leq T$, with exercise time T and payoff function $f : \mathbb{R} \rightarrow \mathbb{R}$. The arbitrage free price is given by

$$C(t) = e^{-r(T-t)} \mathbb{E}^Q[f(S(T)) | \mathcal{F}(t)],$$

where the parameter r is the risk free instantaneous interest rate of a bond used as a numéraire and the measure Q is an equivalent martingale measure (i.e, it is a measure equivalent to \mathbb{P} and under which the discounted price process $e^{-rt}S(t)$ is a martingale). In our case, the market is incomplete and there will be an infinity of equivalent martingale measures (denoted EMM's). Among the wide class of the EMM's, Nicolato and Venardos [53] studied a structure preserving subclass, a subclass under which the log price process and its volatility are again described by a model of the type (7.31). In our setting, we will deal with this structure preserving subclass.

We denote by \mathcal{M} the subset of EMM's such that the log-price process X_ε is still described by a BN-S model. Introduce the following class

$$\mathcal{Y} = \{y : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \int_{\mathbb{R}_+} (\sqrt{y(z)} - 1)^2 \omega(z) dz < \infty\}$$

and for $y \in \mathcal{Y}$, we set

$$\omega^y(z) = y(z)\omega(z). \quad (7.35)$$

Since $\int_{|z| \leq 1} z \omega^y(z) dz < \infty$, we can also define

$$k^y(\theta) = \int_{\mathbb{R}_+} (e^{\theta z} - 1) \omega^y(z) dz, \quad \text{for } \operatorname{Re}(\theta) < 0. \quad (7.36)$$

The following theorem is due to Nicolato and Venardos [53].

Theorem 7.3.1. *Let $y \in \mathcal{Y}$. Then the processes*

$$\psi(t) = \sqrt{Y(t)}^{-1} (r - \mu - (\beta + \frac{1}{2})Y(t) - k^y(\rho))$$

and

$$\psi_\varepsilon(t) = \sqrt{Y_\varepsilon(t)}^{-1} (r - \mu - (\beta + \frac{1}{2})Y_\varepsilon(t) - k^y(\rho)),$$

where k^y is given by (7.36), are such that

$$P\left(\int_0^T \psi^2(s) ds < \infty\right) = 1$$

and

$$P\left(\int_0^T \psi_\varepsilon^2(s) ds < \infty\right) = 1.$$

The processes

$$L^y(t) = \exp\left\{\int_0^t \psi(s) dW(s) - \frac{1}{2} \int_0^t \psi^2(s) ds + \int_0^t \int_0^\infty \log(y(s, z)) N(ds, dz)\right\}$$

$$+ \int_0^t \int_0^\infty (1 - y(s, z)) \omega(z) dz ds \Big\}, \quad 0 \leq t \leq T.$$

and

$$\begin{aligned} L_\varepsilon^y(t) &= \exp \Big\{ \int_0^t \psi_\varepsilon(s) dW(s) - \frac{1}{2} \int_0^t \psi_\varepsilon^2(s) ds + \int_0^t \int_0^\infty \log(y(s, z)) N(ds, dz) \\ &+ \int_0^t \int_0^\infty (1 - y(s, z)) \omega(z) dz ds \Big\}, \quad 0 \leq t \leq T. \end{aligned}$$

are density processes. The probability measures defined by

$$dQ_\varepsilon^y = L(T) d\mathbb{P}$$

and

$$dQ_\varepsilon^y = L_\varepsilon(T) d\mathbb{P}$$

are EMM and the dynamic of X under Q^y is given by

$$\begin{cases} dX(t) &= (r - k^y(\rho) - \frac{1}{2}Y(t))dt + \sqrt{Y}(t)dW^y(t) + \rho dZ(t) \\ dY(t) &= -\lambda Y(t)dt + dZ(t), \quad Y(0) > 0, \end{cases} \quad (7.37)$$

where $W^y(t) = W(t) - \int_0^t \psi(s)ds$ is a Q^y -Brownian motion and $Z(t)$ is a Q^y -Lévy process. The processes W^y and Z are independent under Q^y . The dynamic of X_ε under Q_ε^y is given by

$$\begin{cases} dX_\varepsilon(t) &= (r - k^y(\rho) - \frac{1}{2}Y_\varepsilon(t))dt + \sqrt{Y_\varepsilon}(t)dW_\varepsilon^y(t) + \rho dZ(t) \\ dY_\varepsilon(t) &= -\lambda_\varepsilon Y_\varepsilon(t)dt + dZ(t), \quad Y_\varepsilon(0) > 0, \end{cases} \quad (7.38)$$

where $W_\varepsilon^y(t) = W(t) - \int_0^t \psi_\varepsilon(s)ds$ is a Q_ε^y -Brownian motion, and $Z(t)$ is a Q_ε^y -Lévy process. Z_1 has Lévy density $\omega(z)$ and cumulant transform $k^y(\theta)$ respectively given by (7.35) and (7.36) and the processes W_ε^y and Z are independent under Q_ε^y . Hence $Q^y, Q_\varepsilon^y \in \mathcal{M}$.

In the following lemma, we study the robustness of the dynamic of X under the new measure Q^y .

Lemma 7.3.2. *The system of equation (7.38) converges to (7.37) almost surely when ε goes to 0.*

Proof. For $0 < \lambda_\varepsilon < a$, we have $|Y_\varepsilon(t)| \geq e^{-at}(Z(0) + Z(t))$. Therefore $|\frac{1}{\sqrt{Y_\varepsilon}}| \leq K(t, \omega)$, where $K(t, \omega) = \left(\frac{1}{e^{-at}(Z(0) + Z(t))}\right)^{1/2}$. As $|\frac{1}{\sqrt{Y_\varepsilon}}|$ is defined and continuous for $Y_\varepsilon > 0$, then $\frac{1}{\sqrt{Y_\varepsilon}}$ converges to $\frac{1}{\sqrt{Y}}$ almost surely when ε goes to 0. Therefore ψ_ε converges to ψ when ε goes to 0 and $|\psi_\varepsilon| \leq C(t, \omega)$, where C is a constant depending on time t . By dominated convergence, taking the limit inside the integral in the following expression $W_\varepsilon^y(t) = W(t) - \int_0^t \psi_\varepsilon(s)ds$, we have the convergence of $W_\varepsilon^y(t)$ to $W^y(t)$. Then following the steps of the proof of Lemma 7.3.1, we get the result. \square

In the following, we study the convergence of the option price under the risk-neutral equivalent martingale measure Q_ε^y . Consider the price process $S_\varepsilon(t) = e^{X_\varepsilon(t)}$, $0 \leq t \leq T$. Let $\mathcal{F}_\varepsilon(t)$ be the filtration generated by the Brownian motion $W_\varepsilon^y(t)$ and the Lévy process $Z(t)$. The option price is given by

$$C_\varepsilon(t) = e^{-r(T-t)} \mathbb{E}^{Q_\varepsilon}[f(S_\varepsilon(T)) | \mathcal{F}_\varepsilon(t)].$$

To evaluate the latter expression, we use a Fourier transform approach which extend the method considered in the paper by Nicolato and Venardos [53]. In their approach, they have some restrictions on the Lévy measure which we don't need to consider. We first, state the following results.

The integrated variance over the time period $[t, T]$, is given by $\sigma_\varepsilon^{*2}(t, T) = \int_t^T Y_\varepsilon(s) ds$ and a simple computation shows that

$$\sigma_\varepsilon^{*2}(t, T) = \lambda_\varepsilon^{-1}(1 - e^{-\lambda_\varepsilon(T-t)})Y_\varepsilon(t) + \int_t^T \lambda_\varepsilon^{-1}(1 - e^{-\lambda_\varepsilon(T-s)})dZ(s).$$

Using the Key formula in Eberlein and Raible [32], the Fourier transform of the conditional integrated variance $\sigma_\varepsilon^{*2}(t, T)$ is computed as

$$\mathbb{E}^{Q_\varepsilon}[\exp\{-iu\sigma_\varepsilon^{*2}(t, T)\} | \mathcal{F}_\varepsilon(t)] = \exp\left\{-iuY_\varepsilon(t)\epsilon_\varepsilon(t, T) + \int_t^T k(-iu\epsilon_\varepsilon(s, T))ds\right\},$$

where $\epsilon_\varepsilon(s, T) = \lambda_\varepsilon^{-1}(1 - e^{-\lambda_\varepsilon(T-s)})$. Due to the Theorem 2.2 in Nicolato and Venardos [53], the Fourier transform of the log-price X_ε given the information up to the time $t \leq T$ is given by

$$\begin{aligned} \phi_\varepsilon(u) &= \mathbb{E}^{Q_\varepsilon}[\exp\{-iuX_\varepsilon(T)\} | \mathcal{F}_\varepsilon(t)] \\ &= \exp\left\{-iu(X_\varepsilon(t) + r(T-t) - k^y(\rho)(T-t)) + \frac{1}{2}(-iu - u^2)Y_\varepsilon(t)\epsilon_\varepsilon(t, T) \right. \\ &\quad \left. + \int_t^T k(h_\varepsilon(s, z))ds\right\}, \end{aligned}$$

where $h_\varepsilon(s, z) = -iup + \frac{1}{2}(-iu - u^2)\epsilon_\varepsilon(s, T)$. Notice that, for all $u \in \mathbb{R}$, $\phi_\varepsilon(u)$ converges to $\phi(u)$ when ε goes to 0.

Assume $f, \hat{f} \in L^1(\mathbb{R})$. Then the option price is given by

$$\begin{aligned} C_\varepsilon(t) &= e^{-r(T-t)} \mathbb{E}^{Q_\varepsilon}[f(X_\varepsilon(T)) | \mathcal{F}_\varepsilon(t)] \\ &= e^{-r(T-t)} \frac{1}{2\pi} \mathbb{E}^{Q_\varepsilon}\left[\int_{\mathbb{R}} e^{-iuX_\varepsilon(T)} \hat{f}(y) dy | \mathcal{F}_\varepsilon(t)\right] \\ &= e^{-r(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) \mathbb{E}^{Q_\varepsilon}[e^{-iuX_\varepsilon(T)} | \mathcal{F}_\varepsilon(t)] dy \\ &= e^{-r(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) \phi_\varepsilon(y) dy. \end{aligned} \tag{7.39}$$

Lemma 7.3.3. *The option price $C_\varepsilon(t)$ converges to $C(t)$ when ε goes to 0.*

Proof. From Jensen inequality for the conditional expectation, we have

$$|\phi_\varepsilon(y)| = |\mathbb{E}^{Q_\varepsilon}[\exp\{-iuX_\varepsilon(T)\}|\mathcal{F}_\varepsilon(t)]| \leq \mathbb{E}^{Q_\varepsilon}[|\exp\{-iuX_\varepsilon(T)\}||\mathcal{F}_\varepsilon(t)] \leq 1.$$

Therefore by dominated convergence, we can take the limit inside the integral in (7.39) and the result follows. \square

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